Formal Verification of Avionics Software

JN APMEP

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based on slides by © A. Miné, X. Rival, and P. Cousot
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Outline

1. Introduction
2. Operational semantics
3. Denotational semantics
4. Axiomatic semantics
5. Abstract interpretation
6. Industrial state-of-the-use at Airbus

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Avionics Software for cockpit avionics computers

functions

1. **Aircraft Control Domain**: flight and embedded control systems (**C**, asm)
2. **Airline Information Services Domain**: administrative functions, flight support, and maintenance support (Java)

platforms

1. LynxOS®-based *POSIX* Host Platform for ground-board datalink applications
2. *ARINC 653* Integrated Modular Avionics
3. PikeOS®-based Avionics Server Function

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Motivation for formal verification

- Safety-critical avionics software, certified wrt DO-178 standard
  $\Rightarrow$ need for strong guarantees
- Exponential increase of size & complexity
  $\Rightarrow$ verification costs do not scale
- From process- to product-based assurance

$\Rightarrow$ Transfer of formal verification tools from academic research to operational development teams
15-year collaboration with ENS, CEA, INRIA, AbsInt

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Principle of formal verification

Define the semantics of your program

semantics $\equiv$ mathematical model of the set
of all its possible behaviours in all possible environments

can be constructed from semantics of commands
of the programming language

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Principle of formal verification

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of the programming language

Define a specification

specification ≡ subset of possible behaviours

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Principle of formal verification

Define the semantics of your program

semantics $\equiv$ mathematical model of the set
of all its possible behaviours in all possible environments

can be constructed from semantics of commands
of the programming language

Define a specification

specification $\equiv$ subset of possible behaviours

Conduct a formal proof

that the semantics meets the specification

use computers to automate the proof

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semantics of program P

$Semantics[|P|]$

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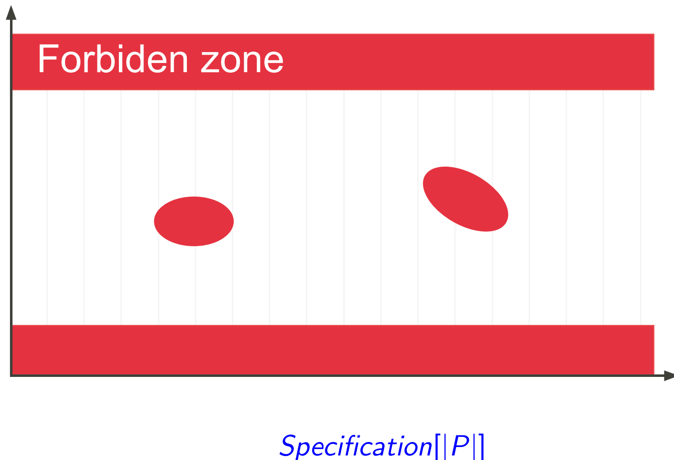
Specification of P (e.g. safety property)



$Specification[|P|]$

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Formal proof of P

$Semantics[|P|] \subseteq Specification[|P|]$

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Excluded miracle

Undecidability

The semantics of a program is not computable.

$\Rightarrow$ Most questions on program behaviour are undecidable.

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Excluded miracle

Undecidability

The semantics of a program is not computable.

$\Rightarrow$ Most questions on program behaviour are undecidable.

Example: termination is undecidable

- assume `termination(P)` always terminates and returns `true` iff `P` always terminates on all input data
- the following program yields a contradiction

```
P := while termination(P)do ()done
```

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Test/simulation

Subset of the possible behaviours ⇒ incomplete

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Operational semantics

Operational semantics

Mathematical description of the execution of programs

1. a **model** of programs: **transition systems**
   - definition, a **small step semantics**
   - example: a simple imperative language
2. **trace semantics:** a families of **big step** semantics
   - **finite** and **infinite** executions
   - **fixpoint**-based definitions

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Definition

We will characterize a program by:

- **states**: photography of the program status at an instant of the execution
- **execution steps**: how do we move from one state to the next one

Definition: transition systems (TS)

A **transition system** is a tuple $(\mathbb{S}, \rightarrow)$ where:

- $\mathbb{S}$ is the **set of states** of the system
- $\rightarrow \subseteq \mathcal{P}(\mathbb{S} \times \mathbb{S})$ is the **transition relation** of the system

Note:

- the set of states **may be infinite**

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A simple imperative language: syntax

We now look at a more classical **imperative language** (intuitively, a bare-bone subset of C):

- **variables** $\mathbb{X}$: finite, predefined set of variables
- **labels** $\mathbb{L}$: before and after each statement
- **values** $\mathbb{V}$: $\mathbb{V}_{\text{int}} \cup \mathbb{V}_{\text{float}} \cup \ldots$

Syntax

e	$::=$	$v \in \mathbb{V}_{\text{int}} \cup \mathbb{V}_{\text{float}} \cup \ldots \mid \mathtt{e} + \mathtt{e} \mid \mathtt{e} * \mathtt{e} \mid \ldots$	expressions
c	$::=$	$\mathtt{TRUE} \mid \mathtt{FALSE} \mid \mathtt{e} < \mathtt{e} \mid \mathtt{e} = \mathtt{e}$	conditions
i	$::=$	$x := \mathtt{e};$	assignment
	$\mid$	**if**(c) b **else** b	condition
	$\mid$	**while**(c) b	loop
b	$::=$	$\{\mathtt{i}; \ldots; \mathtt{i};\}$	block, program($\mathbb{P}$)

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A simple imperative language: states

A **non-error state** should fully describe the configuration at one instant of the program execution:

- the **memory state** defines the current contents of the memory

$$m \in \mathbb{M} = \mathbb{X} \longrightarrow \mathbb{V}$$

- the **control state** defines *where* the program currently is
  - analoguous to the program counter
  - can be defined by adding **labels** $\mathbb{L} = \{\ell_0, \ell_1, \ldots\}$ between each pair of consecutive statements; then:

$$\mathbb{S} = \mathbb{L} \times \mathbb{M} \uplus \{\Omega\}$$

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A simple imperative language: semantics of expressions

- The **semantics $[\![e]\!]$ of expression** e should evaluate each expression into a value, given a memory state
- **Evaluation errors** may occur: division by zero...
  error value is also noted Ω

Thus: $[\![e]\!] : \mathbb{M} \longrightarrow \mathbb{V} \uplus \{\Omega\}$

Definition, by **induction over the syntax:**

$$\begin{array}{rcl} [\![v]\!](m) & = & v \\ [\![x]\!](m) & = & m(x) \\ [\![e_0 + e_1]\!](m) & = & [\![e_0]\!](m) \underline{+} [\![e_1]\!](m) \\ [\![e_0 / e_1]\!](m) & = & \begin{cases} \Omega & \text{if } [\![e_1]\!](m) = 0 \\ [\![e_0]\!](m) \underline{/} [\![e_1]\!](m) & \text{otherwise} \end{cases} \end{array}$$

where $\underline{\oplus}$ is the machine implementation of operator $\oplus$, and is Ω-strict, i.e., $\forall v \in \mathbb{V},\ v \underline{\oplus} \Omega = \Omega \underline{\oplus} v = \Omega$.

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A simple imperative language: semantics of conditions

- The **semantics $[\![\mathtt{c}]\!]$ of condition** $\mathtt{c}$ should return a *boolean value*
- It follows a similar definition to that of the semantics of expressions: $[\![\mathtt{c}]\!] : \mathbb{M} \longrightarrow \mathbb{V}_{\text{bool}} \uplus \{\Omega\}$

Definition, by **induction over the syntax**:

$$
\begin{array}{rcl}
[\![\mathtt{TRUE}]\!](m) & = & \mathtt{TRUE} \\
[\![\mathtt{FALSE}]\!](m) & = & \mathtt{FALSE} \\
[\![\mathtt{e}_0 < \mathtt{e}_1]\!](m) & = & \begin{cases} \mathtt{TRUE} & \text{if } [\![\mathtt{e}_0]\!](m) < [\![\mathtt{e}_1]\!](m) \\ \mathtt{FALSE} & \text{if } [\![\mathtt{e}_0]\!](m) \geq [\![\mathtt{e}_1]\!](m) \\ \Omega & \text{if } [\![\mathtt{e}_0]\!](m) = \Omega \text{ or } [\![\mathtt{e}_1]\!](m) = \Omega \end{cases} \\
[\![\mathtt{e}_0 = \mathtt{e}_1]\!](m) & = & \begin{cases} \mathtt{TRUE} & \text{if } [\![\mathtt{e}_0]\!](m) = [\![\mathtt{e}_1]\!](m) \\ \mathtt{FALSE} & \text{if } [\![\mathtt{e}_0]\!](m) \neq [\![\mathtt{e}_1]\!](m) \\ \Omega & \text{if } [\![\mathtt{e}_0]\!](m) = \Omega \text{ or } [\![\mathtt{e}_1]\!](m) = \Omega \end{cases}
\end{array}
$$

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A simple imperative language: transitions

We now consider the transition induced by each statement.

Case of **assignment** $l_0 : x = \mathrm{e};\ l_1$

- if $[\![\mathrm{e}]\!](m) \neq \Omega$, then $(l_0, m) \to (l_1, m[x \leftarrow [\![\mathrm{e}]\!](m)])$
- if $[\![\mathrm{e}]\!](m) = \Omega$, then $(l_0, m) \to \Omega$

Case of **condition** $l_0 : \textbf{if}(\mathrm{c})\{l_1 : \mathrm{b}_t\ l_2\}\ \textbf{else}\{l_3 : \mathrm{b}_f\ l_4\}\ l_5$

- if $[\![\mathrm{c}]\!](m) = \mathtt{TRUE}$, then $(l_0, m) \to (l_1, m)$
- if $[\![\mathrm{c}]\!](m) = \mathtt{FALSE}$, then $(l_0, m) \to (l_3, m)$
- if $[\![\mathrm{c}]\!](m) = \Omega$, then $(l_0, m) \to \Omega$
- $(l_2, m) \to (l_5, m)$
- $(l_4, m) \to (l_5, m)$

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A simple imperative language: transitions

Case of **loop** l_0 : **while**(c){l_1 : $\mathtt{b}_t$ l_2} l_3

- if $[\![\mathtt{c}]\!](m) = \mathtt{TRUE}$, then $\begin{cases} (l_0, m) \rightarrow (l_1, m) \\ (l_2, m) \rightarrow (l_1, m) \end{cases}$
- if $[\![\mathtt{c}]\!](m) = \mathtt{FALSE}$, then $\begin{cases} (l_0, m) \rightarrow (l_3, m) \\ (l_2, m) \rightarrow (l_3, m) \end{cases}$
- if $[\![\mathtt{c}]\!](m) = \Omega$, then $\begin{cases} (l_0, m) \rightarrow \Omega \\ (l_2, m) \rightarrow \Omega \end{cases}$

Case of $\{l_0 : \mathtt{i}_0; l_1 : \ldots; l_{n-1}\mathtt{i}_{n-1}; l_n\}$

- the transition relation is defined by the individual instructions

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Extending the language with non-determinism

The language we have considered so far is a bit **limited**:

- it is **deterministic**: at most one transition possible from any state
- it does not support the **input of values**

Changes if we model non deterministic inputs...

... with an input instruction:

- $\mathtt{i} ::= \ldots \mid x := \mathbf{input}()$
- $\ell_0 : x := \mathbf{input}(); \ell_1$ generates transitions

$$\forall v \in \mathbb{V},\ (\ell_0, m) \rightarrow (\ell_1, m[x \leftarrow v])$$

- one instruction induces non determinism

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Semantics of real world programming languages

C language:

- several **norms**: ANSI C'99, ANSI C'11, K&R...
- not fully specified:
  - **undefined behavior**
  - **implementation dependent behavior**: architecture (ABI) or implementation (compiler...)
  - unspecified parts: leave room for implementation of compilers and optimizations
- **formalizations** in HOL (C'99), in Coq (CompCert C compiler)

OCaml language:

- more formal...
- ... but still with some unspecified parts, e.g., execution order

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Execution traces

- So far, we considered only states and atomic transitions
- We now consider program **executions** as a whole

Definition: traces

- A **finite trace** is a finite sequence of states $s_0, \ldots, s_n$, noted $\langle s_0, \ldots, s_n \rangle$
- An **infinite trace** is an infinite sequence of states $\langle s_0, \ldots \rangle$

Besides, we write:

- $\mathbb{S}^\star$ for the **set of finite traces**
- $\mathbb{S}^\omega$ for the **set of infinite traces**
- $\mathbb{S}^\propto = \mathbb{S}^\star \cup \mathbb{S}^\omega$ for the **set of finite or infinite traces**

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Semantics of finite traces

We consider a transition system $\mathcal{S} = (\mathbb{S}, \rightarrow)$

Definition

The **finite traces semantics** $[\![\mathcal{S}]\!]^\star$ is defined by:

$$[\![\mathcal{S}]\!]^\star = \{\langle s_0, \dots, s_n \rangle \in \mathbb{S}^\star \mid \forall i,\ s_i \rightarrow s_{i+1}\}$$

Example:

- contrived transition system $\mathcal{S} = (\{a, b, c, d\}, \{(a, b), (b, a), (b, c)\})$
- finite traces semantics:

$$\begin{array}{rcll} [\![\mathcal{S}]\!]^\star & = & \{ & \langle a, b, \dots, a, b, a \rangle, \quad \langle b, a, \dots, a, b, a \rangle, \\ & & & \langle a, b, \dots, a, b, a, b \rangle, \quad \langle b, a, \dots, a, b, a, b \rangle, \\ & & & \langle a, b, \dots, a, b, a, b, c \rangle, \quad \langle b, a, \dots, a, b, a, b, c \rangle \\ & & & \langle c \rangle, \quad \langle d \rangle \quad \} \end{array}$$

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Example: imperative program

Similarly, we can write the traces of a simple imperative program:

$$
\begin{array}{ll}
\ell_0: & \mathrm{x} := 1; \\
\ell_1: & \mathrm{y} := 0; \\
\ell_2: & \textbf{while}(\mathrm{x} < 4)\{ \\
\ell_3: & \quad \mathrm{y} := \mathrm{y} + \mathrm{x}; \\
\ell_4: & \quad \mathrm{x} := \mathrm{x} + 1; \\
\ell_5: & \} \\
\ell_6: & \text{(final program point)}
\end{array}
$$

$$
\begin{array}{rcl}
\tau & = & \langle \; (\ell_0, (\!|\mathrm{x} = \mathrm{x}_0, \mathrm{y} = \mathrm{y}_0|\!)), (\ell_1, (\!|\mathrm{x} = 1, \mathrm{y} = \mathrm{y}_0|\!) \\
& & (\ell_2, (\!|\mathrm{x} = 1, \mathrm{y} = 0|\!)), (\ell_3, (\!|\mathrm{x} = 1, \mathrm{y} = 0|\!)), \\
& & (\ell_4, (\!|\mathrm{x} = 1, \mathrm{y} = 1|\!)), (\ell_5, (\!|\mathrm{x} = 2, \mathrm{y} = 1|\!)), \\
& & (\ell_3, (\!|\mathrm{x} = 2, \mathrm{y} = 1|\!)), (\ell_4, (\!|\mathrm{x} = 2, \mathrm{y} = 3|\!)), \\
& & (\ell_5, (\!|\mathrm{x} = 3, \mathrm{y} = 3|\!)), (\ell_3, (\!|\mathrm{x} = 3, \mathrm{y} = 3|\!)), \\
& & (\ell_4, (\!|\mathrm{x} = 3, \mathrm{y} = 6|\!)), (\ell_5, (\!|\mathrm{x} = 4, \mathrm{y} = 6|\!)), \\
& & (\ell_6, (\!|\mathrm{x} = 4, \mathrm{y} = 6|\!)) \quad \rangle
\end{array}
$$

- very **precise** description of what the program does...
- ... but **quite cumbersome**

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Trace semantics fixpoint form

We define a **semantic function**, that computes the traces of length $i+1$ from the traces of length i (where $i \geq 1$):

Finite traces semantics as a fixpoint

Let $\mathcal{I} = \{\epsilon\} \uplus \{\langle s \rangle \mid s \in \mathbb{S}\}$.
Let $F_\star$ be the function defined by:

$$\begin{array}{rlcl} F_\star : & \mathcal{P}(\mathbb{S}^\star) & \longrightarrow & \mathcal{P}(\mathbb{S}^\star) \\ & X & \longmapsto & X \cup \{\langle s_0, \ldots, s_n, s_{n+1} \rangle \mid \langle s_0, \ldots, s_n \rangle \in X \wedge s_n \to s_{n+1} \end{array}$$

Then, $F_\star$ is **continuous** and thus has a least-fixpoint greater than $\mathcal{I}$; moreover:

$$\mathrm{lfp}_{\mathcal{I}}\, F_\star = [\![\mathcal{S}]\!]^\star = \bigcup_{n\in\mathbb{N}} F_\star^n(\mathcal{I})$$

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Trace semantics fixpoint form: example

Example, with the same simple transition system $\mathcal{S} = (\mathbb{S}, \rightarrow)$:

- $\mathbb{S} = \{a, b, c, d\}$
- $\rightarrow$ is defined by $a \rightarrow b$, $b \rightarrow a$ and $b \rightarrow c$

Then, the first iterates are:

$$
\begin{array}{rcl}
F_\star^0(\mathcal{I}) & = & \{\epsilon, \langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle\} \\
F_\star^1(\mathcal{I}) & = & F_\star^0(\mathcal{I}) \cup \{\langle b, a \rangle, \langle a, b \rangle, \langle b, c \rangle\} \\
F_\star^2(\mathcal{I}) & = & F_\star^1(\mathcal{I}) \cup \{\langle a, b, a \rangle, \langle b, a, b \rangle, \langle a, b, c \rangle\} \\
F_\star^3(\mathcal{I}) & = & F_\star^2(\mathcal{I}) \cup \{\langle b, a, b, a \rangle, \langle a, b, a, b \rangle, \langle b, a, b, c \rangle\} \\
F_\star^4(\mathcal{I}) & = & F_\star^3(\mathcal{I}) \cup \{\langle a, b, a, b, a \rangle, \langle b, a, b, a, b \rangle, \langle a, b, a, b, c \rangle\} \\
F_\star^5(\mathcal{I}) & = & \ldots
\end{array}
$$

- the traces of $[\![\mathcal{S}]\!]^\star$ of length $n + 1$ appear in $F_\star^n(\mathcal{I})$

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Summary

We have discussed:

- **small-step / structural operational semantics:** individual program steps
- **big-step / natural semantics:** program executions as sequences of transitions
- their **fixpoint definitions** and properties

Next:

- another family of semantics, **more compact** and **compositional**
- **semantic program** and **proof methods**

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Introduction

Operational semantics

Defined as small execution steps *(transition relation)*
over low-level internal configurations *(states)*

Transitions are chained to define *(maximal)* traces
possibly abstracted as input-output relations *(big-step)*

Denotational semantics

Direct functions from programs to mathematical objects *(denotations)*
by induction on the program syntax *(compositional)*
ignoring intermediate steps and execution details *(no state)*

$\Longrightarrow$ Higher-level, more abstract, more modular.
Tries to decouple a program meaning from its execution.
Focus on the mathematical structures that represent programs.
(founded by Strachey and Scott in the 70s: [Scott-Strachey71])

"Assembly" of semantics vs. "Functional programming" of semantics

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Denotation worlds

- **imperative programs**

  effect of a program: mutate a memory state
  natural denotation: input/output function
  $\mathcal{D} \simeq \textit{memory} \to \textit{memory}$

  challenge: build a whole program denotation
  from denotations of atomic language constructs (modularity)

$\Longrightarrow$ very rich theory of mathematical structures
(Scott domains, cartesian closed categories, coherent spaces, event structures, game semantics, etc. We will not present them in this overview!)

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A simple imperative language: IMP

IMP expressions

$expr$	$::=$	X	(*variable*)
	$\mid$	c	(*constant*)
	$\mid$	$\diamond\ expr$	(*unary operation*)
	$\mid$	$expr \diamond expr$	(*binary operation*)

- variables in a fixed set $X \in \mathcal{V}$
- constants $\mathcal{I} \stackrel{\text{def}}{=} \mathbb{B} \cup \mathbb{Z}$:
  - booleans $\mathbb{B} \stackrel{\text{def}}{=} \{\,\text{true}, \text{false}\,\}$
  - integers $\mathbb{Z}$
- operations $\diamond$:
  - integer operations: $+$, $-$, $\times$, $/$, $<$, $\leq$
  - boolean operations: $\neg$, $\wedge$, $\vee$
  - polymorphic operations: $=$, $\neq$

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A simple imperative language: IMP

Statements

stat	::=	**skip**	(*do nothing*)
	|	$X \leftarrow expr$	(*assignment*)
	|	*stat*; *stat*	(*sequence*)
	|	**if** *expr* **then** *stat* **else** *stat*	(*conditional*)
	|	**while** *expr* **do** *stat*	(*loop*)

(inspired from the presentation in [Benton96])

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Expression semantics

$\mathbb{E}[\![\, expr \,]\!] : \mathcal{E} \rightharpoonup \mathcal{I}$

- environments $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{V} \to \mathcal{I}$ map variables in $\mathcal{V}$ to values in $\mathcal{I}$
- $\mathbb{E}[\![\, expr \,]\!]$ returns a value in $\mathcal{I}$
- $\rightharpoonup$ denotes partial functions (as opposed to $\to$)
  necessary because some operations are undefined
  - $1 + \text{true}$, $1 \wedge 2$ (type mismatch)
  - $3/0$ (invalid value)
- defined by structural induction on abstract syntax trees

  (next slide)

(when we use the notation $\mathbb{X}[\![\, y \,]\!]$, y is a syntactic object; X serves to distinguish between different semantic functions with different signatures, often varying with the kind of syntactic object y (expression, statement, etc.);
$\mathbb{X}[\![\, y \,]\!]z$ is the application of the function $\mathbb{X}[\![\, y \,]\!]$ to the object z)

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Expression semantics

$\mathbb{E}[\![\, expr \,]\!] : \mathcal{E} \rightharpoonup \mathcal{I}$

$$
\begin{array}{llll}
\mathbb{E}[\![c]\!]\rho & \stackrel{\text{def}}{=} & c & \in \mathcal{I} \\
\mathbb{E}[\![V]\!]\rho & \stackrel{\text{def}}{=} & \rho(V) & \in \mathcal{I} \\
\mathbb{E}[\![-e]\!]\rho & \stackrel{\text{def}}{=} & -v & \in \mathbb{Z} \quad \text{if } v = \mathbb{E}[\![e]\!]\rho \in \mathbb{Z} \\
\mathbb{E}[\![\neg e]\!]\rho & \stackrel{\text{def}}{=} & \neg v & \in \mathbb{B} \quad \text{if } v = \mathbb{E}[\![e]\!]\rho \in \mathbb{B} \\
\mathbb{E}[\![e_1 + e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 + v_2 & \in \mathbb{Z} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{Z}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{Z} \\
\mathbb{E}[\![e_1 - e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 - v_2 & \in \mathbb{Z} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{Z}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{Z} \\
\mathbb{E}[\![e_1 \times e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 \times v_2 & \in \mathbb{Z} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{Z}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{Z} \\
\mathbb{E}[\![e_1 / e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 / v_2 & \in \mathbb{Z} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{Z}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{Z} \setminus \{0\} \\
\mathbb{E}[\![e_1 \wedge e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 \wedge v_2 & \in \mathbb{B} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{B}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{B} \\
\mathbb{E}[\![e_1 \vee e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 \vee v_2 & \in \mathbb{B} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{B}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{B} \\
\mathbb{E}[\![e_1 < e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 < v_2 & \in \mathbb{B} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{Z}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{Z} \\
\mathbb{E}[\![e_1 \leq e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 \leq v_2 & \in \mathbb{B} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathbb{Z}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathbb{Z} \\
\mathbb{E}[\![e_1 = e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 = v_2 & \in \mathbb{B} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathcal{I}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathcal{I} \\
\mathbb{E}[\![e_1 \neq e_2]\!]\rho & \stackrel{\text{def}}{=} & v_1 \neq v_2 & \in \mathbb{B} \quad \text{if } v_1 = \mathbb{E}[\![e_1]\!]\rho \in \mathcal{I}, v_2 = \mathbb{E}[\![e_2]\!]\rho \in \mathcal{I}
\end{array}
$$

undefined otherwise

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Statement semantics

$\mathbb{S}[\![\, stat \,]\!] : \mathcal{E} \rightharpoonup \mathcal{E}$

- maps an environment before the statement to an environment after the statement
- partial function due to
  - errors in expressions
  - non-termination
- also defined by structural induction

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Summary

Rewriting the semantics using total functions on cpos:

- $\mathbb{E}[\![\, expr \,]\!] : \mathcal{E}_\perp \xrightarrow{c} \mathcal{I}_\perp$

  returns $\perp$ for an error or if its argument is $\perp$
- $\mathbb{S}[\![\, stat \,]\!] : \mathcal{E}_\perp \xrightarrow{c} \mathcal{E}_\perp$
  - $\mathbb{S}[\![\, \textbf{skip} \,]\!]\rho \stackrel{\text{def}}{=} \rho$
  - $\mathbb{S}[\![\, e_1; e_2 \,]\!] \stackrel{\text{def}}{=} \mathbb{S}[\![\, e_2 \,]\!] \circ \mathbb{S}[\![\, e_1 \,]\!]$
  - $\mathbb{S}[\![\, X \leftarrow e \,]\!]\rho \stackrel{\text{def}}{=} \begin{cases} \perp & \text{if } \mathbb{E}[\![\, e \,]\!]\rho = \perp \\ \rho[X \mapsto \mathbb{E}[\![\, e \,]\!]\rho] & \text{otherwise} \end{cases}$
  - $\mathbb{S}[\![\, \textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \,]\!]\rho \stackrel{\text{def}}{=} \begin{cases} \mathbb{S}[\![\, s_1 \,]\!]\rho & \text{if } \mathbb{E}[\![\, e \,]\!]\rho = \text{true} \\ \mathbb{S}[\![\, s_2 \,]\!]\rho & \text{if } \mathbb{E}[\![\, e \,]\!]\rho = \text{false} \\ \perp & \text{otherwise} \end{cases}$
  - $\mathbb{S}[\![\, \textbf{while } e \textbf{ do } s \,]\!] \stackrel{\text{def}}{=} \text{lfp}\, F$

    where $F(f)(\rho) = \begin{cases} \rho & \text{if } \mathbb{E}[\![\, e \,]\!]\rho = \text{false} \\ f(\mathbb{S}[\![\, s \,]\!]\rho) & \text{if } \mathbb{E}[\![\, e \,]\!]\rho = \text{true} \\ \perp & \text{otherwise} \end{cases}$

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Statement semantics: loops

How do we handle loops?

the semantics of loops must satisfy:

$$\mathbb{S}[\![\, \textbf{while } e \textbf{ do } s \,]\!]\rho = \begin{cases} \rho & \text{if } \mathbb{E}[\![\, e \,]\!]\rho = \text{false} \\ \mathbb{S}[\![\, \textbf{while } e \textbf{ do } s \,]\!](\mathbb{S}[\![\, s \,]\!]\rho) & \text{if } \mathbb{E}[\![\, e \,]\!]\rho = \text{true} \\ \text{undefined} & \text{otherwise} \end{cases}$$

this is a recursive definition, we must prove that:

- the equation has solutions
- choose the right one

$\Longrightarrow$ we use fixpoints on partially ordered sets

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- Deterministic imperative programs
- Link between operational and denotational semantics

4 Axiomatic semantics

5 Abstract interpretation

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Motivation

Are the operational and denotational semantics consistent with each other?

Note that:

- systems are actually described operationally
- the denotational semantics is a more abstract representation (more suitable for some reasoning on the system)

$\Longrightarrow$ the denotational semantics must be proven faithful (in some sense) to the operational model to be of any use

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Reminder: from traces to big-step semantics

Big-step semantics: abstraction of traces
only remembers the input-output relations

many variants exist:

- "angelic" semantics, in $\mathcal{P}(\Sigma \times \Sigma)$:
  $\mathbb{A}[\![s]\!] \stackrel{\text{def}}{=} \{ (\sigma, \sigma') \mid \exists (\sigma_0, \ldots, \sigma_n) \in t[\![s]\!]^* : \sigma = \sigma_0, \sigma' = \sigma_n \}$
  (only give information on the terminating behaviors;
  can only prove partial correctness)
- natural semantics, in $\mathcal{P}(\Sigma \times \Sigma_\perp)$:
  $\mathbb{N}[\![s]\!] \stackrel{\text{def}}{=} \mathbb{A}[\![s]\!] \cup \{ (\sigma, \perp) \mid \exists (\sigma_0, \ldots) \in t[\![s]\!]^\omega : \sigma = \sigma_0 \}$
  (models the terminating and non-terminating behaviors;
  can prove total correctness)

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Semantic diagram

(α is an isomorphism)

denotational world | **operational world**

α

$\mathbb{S}[\![s]\!]$ denotational - - - - - - - - - - - big step $\mathbb{A}[\![s]\!]$

natural $\mathbb{N}[\![s]\!]$

traces $\approx[\![s]\!]$

transition system (small step) $\tau[s]$

statement

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Introduction

Operational semantics

Models precisely program execution as low-level transitions between internal states

(transition systems, execution traces, big-step semantics)

Denotational semantics

Maps programs into objects in a mathematical domain

(higher level, compositional, domain oriented)

Aximoatic semantics

Prove properties about programs

- programs are annotated with logical assertions
- a rule-system defines the validity of assertions (logical proofs)
- clearly separates programs from specifications
  (specification $\simeq$ user-provided abstraction of the behavior, it is not unique)
- enables the use of logic tools (partial automation)

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Example: function specification

example in C + ACSL

```
int mod(int A, int B) {
  int Q = 0;
  int R = A;
  while (R >= B) {
    R = R - B;
    Q = Q + 1;
  }
  return R;
}
```

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Example: function specification

example in C + ACSL

```
//@ ensures \result == A mod B;
int mod(int A, int B) {
  int Q = 0;
  int R = A;
  while (R >= B) {
    R = R - B;
    Q = Q + 1;
  }
  return R;
}
```

- express the intended behavior of the function (returned value)

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Example: function specification

example in C + ACSL

```
//@ requires A>=0 && B>=0;
//@ ensures \result == A mod B;
int mod(int A, int B) {
  int Q = 0;
  int R = A;
  while (R >= B) {
    R = R - B;
    Q = Q + 1;
  }
  return R;
}
```

- express the intended behavior of the function (returned value)
- add requirements for the function to actually behave as intended
  (a requires/ensures pair is a function contract)

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Example: function specification

example in C + ACSL

```
//@ requires A>=0 && B>0;
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int mod(int A, int B) {
  int Q = 0;
  int R = A;
  while (R >= B) {
    R = R - B;
    Q = Q + 1;
  }
  return R;
}
```

- express the intended behavior of the function (returned value)
- add requirements for the function to actually behave as intended (a requires/ensures pair is a function contract)
- strengthen the requirements to ensure termination

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Example: program annotations

example with full assertions

```
//@ requires A>=0 && B>0;
//@ ensures \result == A mod B;
int mod(int A, int B) {
  int Q = 0;
  int R = A;
  //@ assert A>=0 && B>0 && Q=0 && R==A;
  while (R >= B) {
    //@ assert A>=0 && B>0 && R>=B && A==Q*B+R;
    R = R - B;
    Q = Q + 1;
  }
  //@ assert A>=0 && B>0 && R>=0 && R<B && A==Q*B+R;
  return R;
}
```

Assertions give detail about the internal computations
why and how contracts are fulfilled

(Note: $r = a \mod b$ means $\exists q: a = qb + r \wedge 0 \leq r < b$)

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Language support

Contracts (and class invariants):

- built in few languages (Eiffel)
- available as a library / external tool (C, Java, C#, etc.)

Contracts can be:

- checked dynamically
- checked statically (Frama-C, Why, ESC/Java)
- inferred statically (CodeContracts)

In this talk:

deductive methods (logic) to check (prove) statically (at compile-time) partially automatically (with user help) that contracts hold

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Hoare triples

Hoare triple: $\{P\}\ prog\ \{Q\}$

- *prog* is a program fragment
- P and Q are logical assertions over program variables
  (e.g. $P \stackrel{\text{def}}{=} (X \geq 0 \wedge Y \geq 0) \vee (X < 0 \wedge Y < 0)$)

A triple means:

- if P holds before *prog* is executed
- then Q holds after the execution of *prog*
- unless *prog* does not terminates or encounters an error

P is the precondition, Q is the postcondition

$\{P\}\ prog\ \{Q\}$ expresses partial correctness
(does not rule out errors and non-termination)

Hoare triples serve as judgements in a proof system
(introduced in [Hoare69])

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Language

$$
\begin{array}{rcll}
\mathit{stat} & ::= & X \leftarrow \mathit{expr} & (\mathit{assignment}) \\
& | & \textbf{skip} & (\mathit{do\ nothing}) \\
& | & \textbf{fail} & (\mathit{error}) \\
& | & \mathit{stat};\ \mathit{stat} & (\mathit{sequence}) \\
& | & \textbf{if}\ \mathit{expr}\ \textbf{then}\ \mathit{stat}\ \textbf{else}\ \mathit{stat} & (\mathit{conditional}) \\
& | & \textbf{while}\ \mathit{expr}\ \textbf{do}\ \mathit{stat} & (\mathit{loop})
\end{array}
$$

- $X \in \mathcal{V}$: integer-valued variables
- *expr*: integer arithmetic expressions

  we assume that:

  - expressions are deterministic (for now)
  - expression evaluation do not cause error

    for instance, to avoid division by zero, we can:
    either define $1/0$ to be a valid value, such as 0
    or systematically guard divisions
    (e.g.: **if** $X = 0$ **then fail else** $\cdots/X\cdots$)

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Hoare rules: summary

$$\frac{}{\{P\}\ \textbf{skip}\ \{P\}} \qquad \frac{}{\{\text{true}\}\ \textbf{fail}\ \{\text{false}\}} \qquad \frac{}{\{P[e/X]\}\ X \leftarrow e\ \{P\}}$$

$$\frac{\{P\}\ s\ \{R\} \qquad \{R\}\ t\ \{Q\}}{\{P\}\ s; t\ \{Q\}} \qquad \frac{\{P \wedge e\}\ s\ \{Q\} \qquad \{P \wedge \neg e\}\ t\ \{Q\}}{\{P\}\ \textbf{if}\ e\ \textbf{then}\ s\ \textbf{else}\ t\ \{Q\}}$$

$$\frac{\{P \wedge e\}\ s\ \{P\}}{\{P\}\ \textbf{while}\ e\ \textbf{do}\ s\ \{P \wedge \neg e\}}$$

$$\frac{P \Rightarrow P' \qquad Q' \Rightarrow Q \qquad \{P'\}\ c\ \{Q'\}}{\{P\}\ c\ \{Q\}}$$

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Proof tree example

$$s \stackrel{\text{def}}{=} \textbf{while } I < N \textbf{ do } (X \leftarrow 2X;\ I \leftarrow I+1)$$

$$\dfrac{A \qquad B \qquad \dfrac{C \qquad \dfrac{}{\{P_3\}\ X \leftarrow 2X\ \{P_2\}} \qquad \dfrac{}{\{P_2\}\ I \leftarrow I+1\ \{P_1\}}}{\dfrac{\{P_1 \wedge I < N\}\ X \leftarrow 2X;\ I \leftarrow I+1\ \{P_1\}}{\{P_1\}\ s\ \{P_1 \wedge I \geq N\}}}}{\{X = 1 \wedge I = 0 \wedge N \geq 0\}\ s\ \{X = 2^N \wedge N = I \wedge N \geq 0\}}$$

$$P_1 \stackrel{\text{def}}{=} X = 2^I \wedge I \leq N \wedge N \geq 0$$

$$P_2 \stackrel{\text{def}}{=} X = 2^{I+1} \wedge I+1 \leq N \wedge N \geq 0$$

$$P_3 \stackrel{\text{def}}{=} 2X = 2^{I+1} \wedge I+1 \leq N \wedge N \geq 0 \quad \equiv X = 2^I \wedge I < N \wedge N \geq 0$$

$A : (X = 1 \wedge I = 0 \wedge N \geq 0) \Rightarrow P_1$

$B : (P_1 \wedge I \geq N) \Rightarrow (X = 2^N \wedge N = I \wedge N \geq 0)$

$C : P_3 \iff (P_1 \wedge I < N)$

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Soundness and completeness

Validity:

$\{P\}\ c\ \{Q\}$ is valid $\stackrel{\text{def}}{\iff}$ executions starting in a state satisfying P
and terminating
end in a state satisfying Q

(it is an operational notion)

- **soundness**

  a proof tree exists for $\{P\}\ c\ \{Q\} \implies \{P\}\ c\ \{Q\}$ is valid

- **completeness**

  $\{P\}\ c\ \{Q\}$ is valid $\implies$ a proof tree exists for $\{P\}\ c\ \{Q\}$

  (technically, by Gödel's incompleteness theorem, $P \Rightarrow Q$ is not always provable for strong theories; hence, Hoare logic is incomplete; we consider relative completeness by adding all valid properties $P \Rightarrow Q$ on assertions as axioms)

Theorem (Cook 1974)

Hoare logic is sound (and relatively complete)

Completeness no longer holds for more complex languages (Clarke 1976)

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Link with denotational semantics

Reminder: $\mathbb{S}[\![\, stat \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$ where $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{V} \mapsto \mathcal{I}$

$$\mathbb{S}[\![\, \textbf{skip} \,]\!] R \stackrel{\text{def}}{=} R$$

$$\mathbb{S}[\![\, \textbf{fail} \,]\!] R \stackrel{\text{def}}{=} \emptyset$$

$$\mathbb{S}[\![\, s_1 ; s_2 \,]\!] \stackrel{\text{def}}{=} \mathbb{S}[\![\, s_2 \,]\!] \circ \mathbb{S}[\![\, s_1 \,]\!]$$

$$\mathbb{S}[\![\, X \leftarrow e \,]\!] R \stackrel{\text{def}}{=} \{\, \rho[X \mapsto v] \mid \rho \in R,\ v \in \mathbb{E}[\![\, e \,]\!] \rho \,\}$$

$$\mathbb{S}[\![\, \textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \,]\!] R \stackrel{\text{def}}{=} \mathbb{S}[\![\, s_1 \,]\!] \{\, \rho \in R \mid \text{true} \in \mathbb{E}[\![\, e \,]\!] \rho \,\} \cup \mathbb{S}[\![\, s_2 \,]\!] \{\, \rho \in R \mid \text{false} \in \mathbb{E}[\![\, e \,]\!] \rho \,\}$$

$$\mathbb{S}[\![\, \textbf{while } e \textbf{ do } s \,]\!] R \stackrel{\text{def}}{=} \{\, \rho \in \text{lfp } F \mid \text{false} \in \mathbb{E}[\![\, e \,]\!] \rho \,\}$$
$$\text{where } F(X) \stackrel{\text{def}}{=} R \cup \mathbb{S}[\![\, s \,]\!] \{\, \rho \in X \mid \text{true} \in \mathbb{E}[\![\, e \,]\!] \rho \,\}$$

Theorem

$$\{P\}\ c\ \{Q\} \stackrel{\text{def}}{\iff} \forall R \subseteq \mathcal{E}\colon R \models P \implies \mathbb{S}[\![\, c \,]\!] R \models Q$$

($A \models P$ means $\forall \rho \in A$, the formula P is true on the variable assignment ρ)

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Link with denotational semantics

- Hoare logic reasons on formulas
- denotational semantics reasons on state sets

we can assimilate assertion formulas and state sets
(logical abuse: we assimilate formulas and models)

let $[R]$ be any formula representing the set R, then:

- $\{[R]\}\ c\ \{[\mathbb{S}[\![\, c \,]\!] R]\}$ is always valid
- $\{[R]\}\ c\ \{[R']\} \Rightarrow \mathbb{S}[\![\, c \,]\!] R \subseteq R'$

$\Longrightarrow$ $[\mathbb{S}[\![\, c \,]\!] R]$ provides the best valid postcondition

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Dijkstra's weakest liberal preconditions

Principle:

- calculus to derive preconditions from postconditions
- order and mechanize the search for intermediate assertions

  (easier to go backwards, mainly due to assignments)

Weakest liberal precondition $\quad wlp : (prog \times Prop) \rightarrow Prop$

$wlp(c, P)$ is the weakest, i.e. most general, precondition
ensuring that $\{wlp(c, P)\}\ c\ \{P\}$ is a Hoare triple

(greatest state set that ensures that the computation ends up in P)

formally: $\quad \{P\}\ c\ \{Q\} \iff (P \Rightarrow wlp(c, Q))$

"liberal" means that we do not care about termination and errors

Examples:

$wlp(X \leftarrow X + 1,\ X = 1) =$
$wlp(\textbf{while}\ X < 0\ X \leftarrow X + 1,\ X \geq 0) =$
$wlp(\textbf{while}\ X \neq 0\ X \leftarrow X + 1,\ X \geq 0) =$

(introduced in [Dijkstra75])

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Dijkstra's weakest liberal preconditions

Principle:

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(greatest state set that ensures that the computation ends up in P)

formally: $\quad \{P\}\ c\ \{Q\} \iff (P \Rightarrow wlp(c, Q))$

"liberal" means that we do not care about termination and errors

Examples:

$$wlp(X \leftarrow X + 1,\ X = 1) = (X = 0)$$
$$wlp(\textbf{while } X < 0\ X \leftarrow X + 1,\ X \geq 0) = \text{true}$$
$$wlp(\textbf{while } X \neq 0\ X \leftarrow X + 1,\ X \geq 0) = \text{true}$$

(introduced in [Dijkstra75])

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A calculus for wlp

wlp is defined by induction on the syntax of programs:

$$wlp(\textbf{skip}, P) \stackrel{\text{def}}{=} P$$

$$wlp(\textbf{fail}, P) \stackrel{\text{def}}{=} \text{true}$$

$$wlp(X \leftarrow e, P) \stackrel{\text{def}}{=} P[e/X]$$

$$wlp(s; t, P) \stackrel{\text{def}}{=} wlp(s, wlp(t, P))$$

$$wlp(\textbf{if } e \textbf{ then } s \textbf{ else } t, P) \stackrel{\text{def}}{=} (e \Rightarrow wlp(s, P)) \land (\neg e \Rightarrow wlp(t, P))$$

$$wlp(\textbf{while } e \textbf{ do } s, P) \stackrel{\text{def}}{=} I \land ((e \land I) \Rightarrow wlp(s, I)) \land ((\neg e \land I) \Rightarrow P)$$

- $e \Rightarrow Q$ is equivalent to $Q \lor \neg e$
  weakest property that matches Q when e holds
  but says nothing when e does not hold
- **while** loops require providing an invariant predicate I
  intuitively, *wlp* checks that I is an inductive invariant implying P
  if so, it returns I; otherwise, it returns false

wlp is the weakest precondition only if I is well-chosen. . .

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Conclusion

- logic allows us to reason about program correctness
- verification can be reduced to proofs of simple logic statements

Issue: automation

- annotations are required (loop invariants, contracts)
- verification conditions must be proven

to scale up to realistic programs, we need to automate as much as possible

Some solutions:

- automatic logic solvers to discharge proof obligations
  SAT / SMT solvers
- abstract interpretation to approximate the semantics
  - fully automatic
  - able to infer invariants

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Concrete semantics of program P

$x(t)$

t

Possible trajectories

Semantics$[|P|]$

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Specification of P (e.g. safety property)

$Specification[|P|]$

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Formal proof of P

$$Semantics[|P|] \subseteq Specification[|P|]$$

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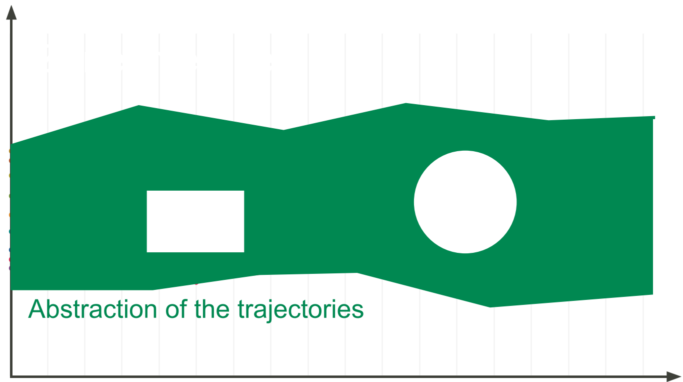
Abstract semantics for P



$Abstraction(Semantics[|P|])$

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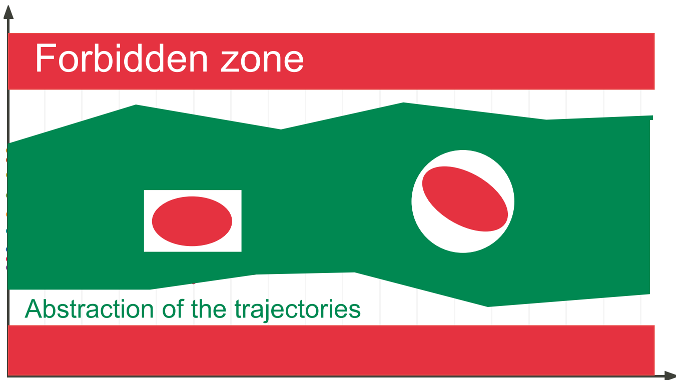
Proof by abstract interpretation



$$Abstraction(Semantics[|P|]) \subseteq Specification[|P|]$$

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Soundness of abstract interpretation

$$Semantics[|P|] \subseteq Abstraction(Semantics[|P|]) \subseteq Specification[|P|]$$

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Formal methods are abstract interpretations

model-checking : user-provided abstract semantics
$\triangleq$ finitary model
may be inferred by static analysis

deductive methods : user-provided abstract semantics
$\triangleq$ inductive invariants
may be inferred by static analysis

static analysis : abstract semantics computed automatically
$\triangleq$ predefined abstractions
may be tailored by the user

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Outline

1 Introduction

2 Operational semantics

3 Denotational semantics

4 Axiomatic semantics

5 Abstract interpretation
- Principle
- Example of abstract interpretation
- Interval analysis, more formally
- Alarms
- Abstract domains

6 Industrial state-of-the-use at Airbus

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Concrete semantics: set of (discrete) traces

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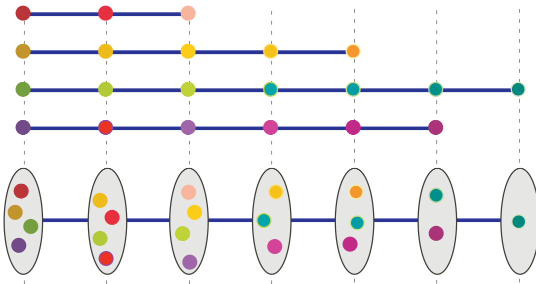
Collecting semantics

Collect the set of states that can appear on some trace at any given discrete time :



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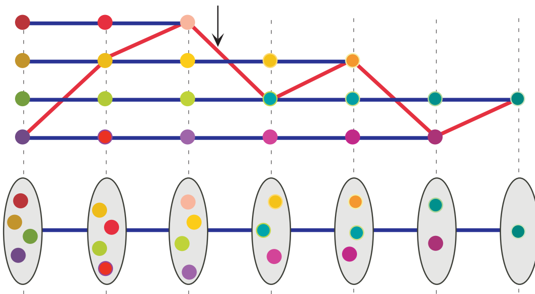
Trace abstraction : collecting abstraction

This an abstraction. Does the red trace exist?

Trace semantics: no $\neq$ collecting semantics: I don't know.



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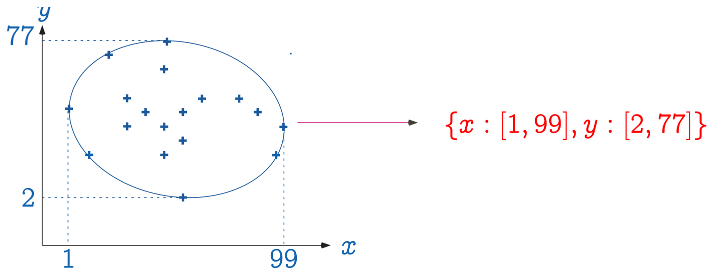
Set abstraction : intervals



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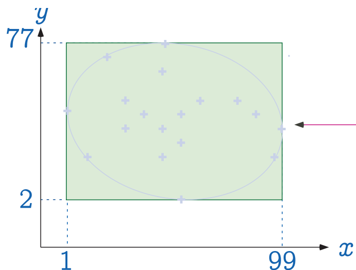
Set abstraction : intervals



$\{x : [1, 99], y : [2, 77]\}$

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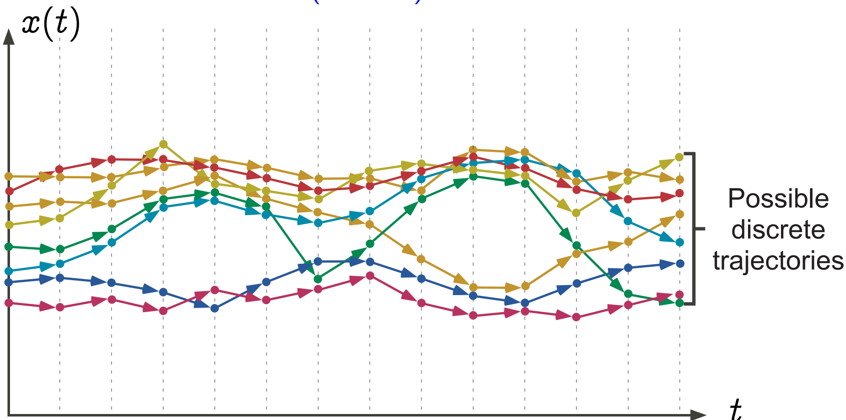
From set of traces to set of reachable states

set of (discrete) traces



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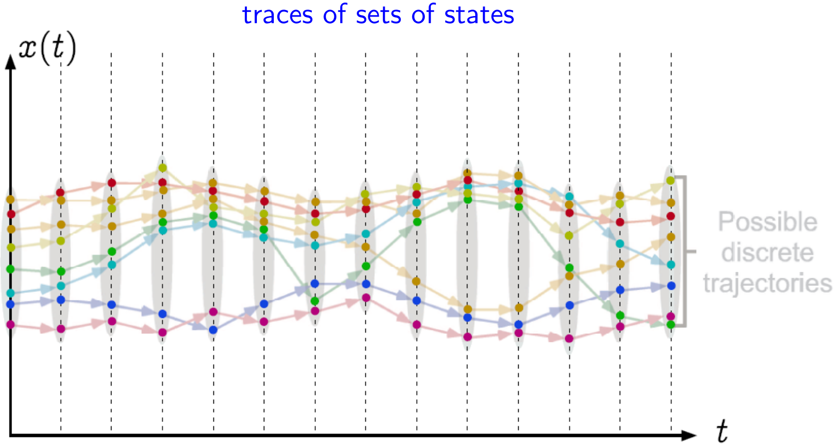
From set of traces to set of reachable states



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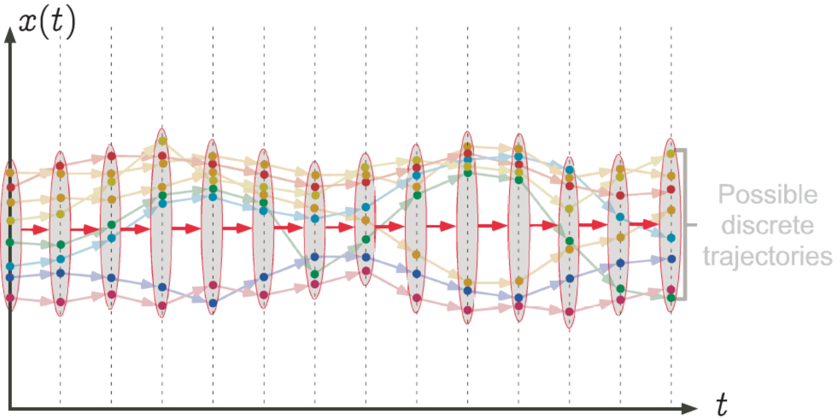
From set of traces to set of reachable states

trace of sets of states



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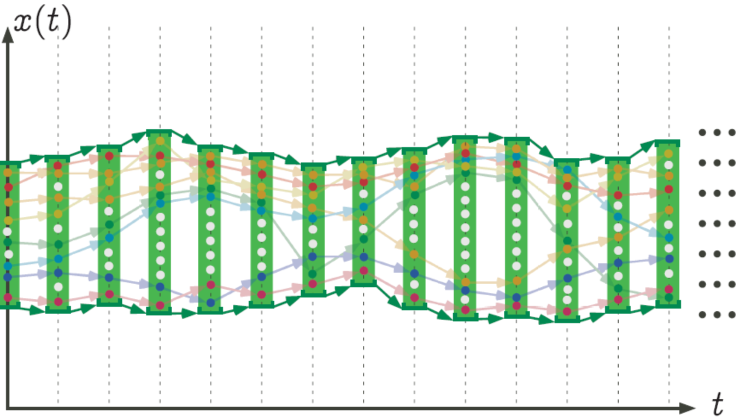
From set of traces to set of reachable states

trace of intervals



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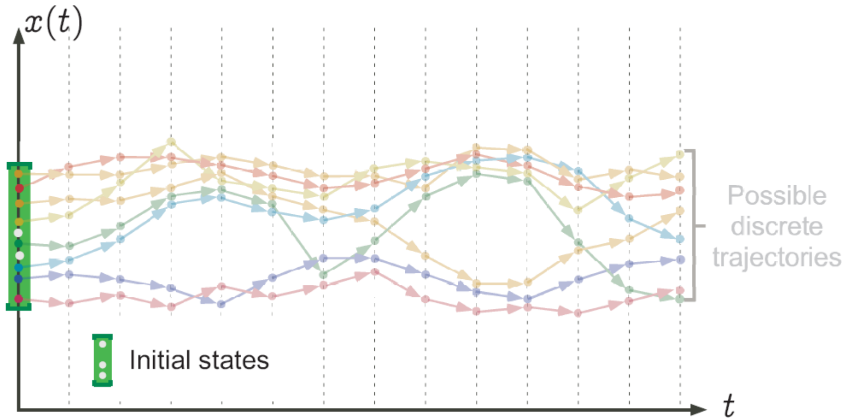
From set of traces to set of reachable states

Effective computation : intialisation



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From set of traces to set of reachable states

Effective computation : propagation

$x(t)$

Possible discrete trajectories

Interval transition

t

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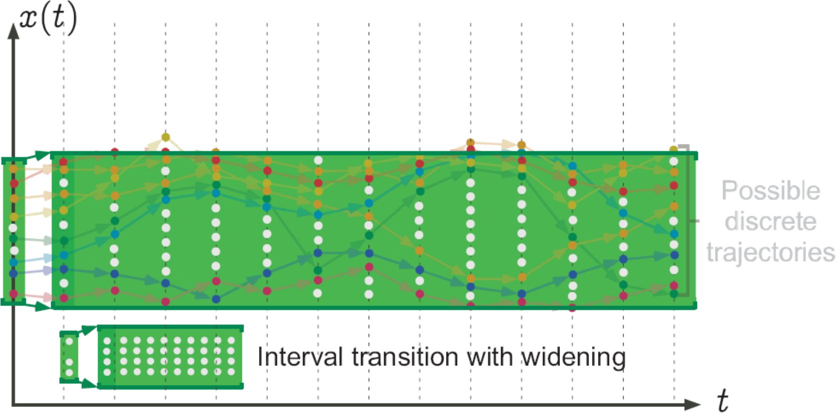
From set of traces to set of reachable states

Effective computation : widening unstable constraints



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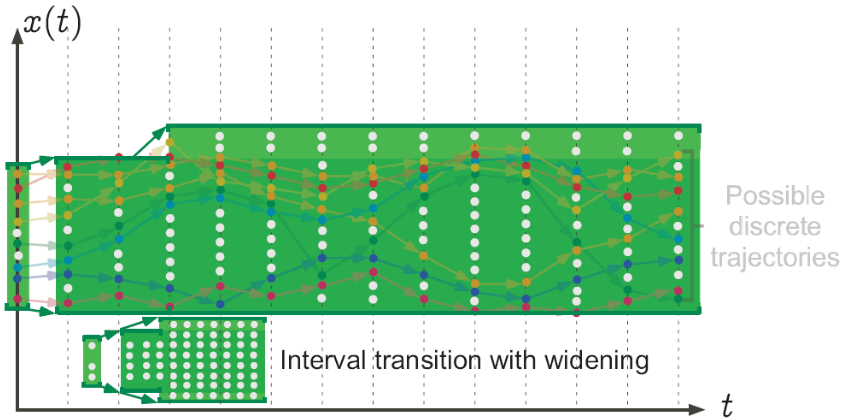
From set of traces to set of reachable states

Effective computation : widening unstable constraints



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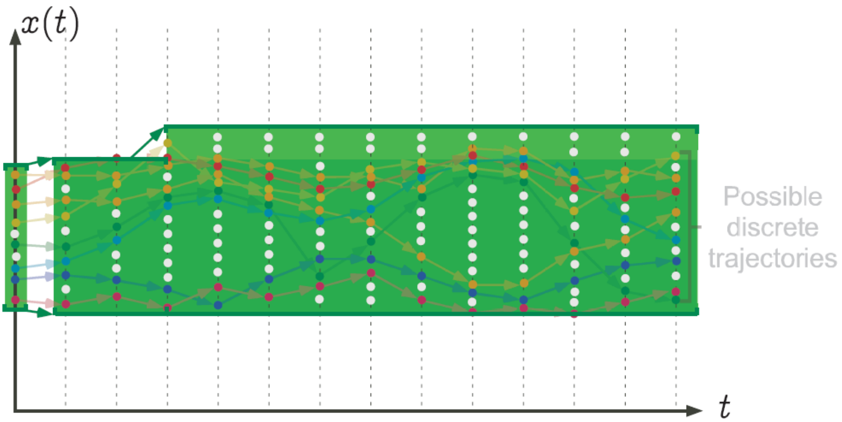
From set of traces to set of reachable states



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Interval analysis

Program to be analyzed

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

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Interval analysis

Equations (abstract interpretation of the semantics)

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$
\begin{cases}
X_1 = [1,1] \\
X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\
X_3 = X_2 \oplus [1,1] \\
X_4 = (X_1 \cup X_3) \cap [10000, +\infty]
\end{cases}
$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = \emptyset \\ X_2 = \emptyset \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = \emptyset \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,1] \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,1] \\ X_3 = [2,2] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
     x := 1;
1:
     while x < 10000 do
2:
          x := x + 1
3:
     od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,2] \\ X_3 = [2,2] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,2] \\ X_3 = [2,3] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
      x := 1;
1:
      while x < 10000 do
2:
            x := x + 1
3:
      od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,3] \\ X_3 = [2,3] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
    x := 1;
1:
    while x < 10000 do
2:
        x := x + 1
3:
    od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,3] \\ X_3 = [2,4] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,4] \\ X_3 = [2,4] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,4] \\ X_3 = [2,5] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
        x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,5] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Resolution by increasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,6] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Convergence speed-up by widening

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,+\infty] & \Leftarrow \text{widening} \\ X_3 = [2,6] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Decreasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, +\infty] \\ X_3 = [2, +\infty] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Decreasing iterations

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +\infty] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Decreasing iterations

```
     x := 1;
1:
     while x < 10000 do
2:
          x := x + 1
3:
     od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = \emptyset \end{cases}$$

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Interval analysis

Final solution

```
        x := 1;
1:
        while x < 10000 do
2:
            x := x + 1
3:
        od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

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Interval analysis

Result of interval analysis

```
   x := 1;
1: {x = 1}
   while x < 10000 do
2: {x ∈ [1, 9999]}
        x := x + 1
3: {x ∈ [2, +10000]}
   od;
4: {x = 10000}
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

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Interval analysis

Formal proof of absence of overflow

```
   x := 1;
1: {x = 1}
   while x < 10000 do
2: {x ∈ [1, 9999]}          ⟵ no overflow
       x := x + 1
3: {x ∈ [2, +10000]}
   od;
4: {x = 10000}
```

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Outline

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6. Industrial state-of-the-use at Airbus

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Language

Expressions and conditions

$$
\begin{array}{llll}
\textit{expr} & ::= & V & V \in \mathcal{V} \\
 & | & c & c \in \mathbb{Z} \\
 & | & -\textit{expr} & \\
 & | & \textit{expr} \diamond \textit{expr} & \diamond \in \{+, -, \times, /\} \\
 & | & \textbf{rand}(a, b) & a, b \in \mathbb{Z} \\
\textit{cond} & ::= & \textit{expr} \bowtie \textit{expr} & \bowtie \in \{\leq, \geq, =, \neq, <, >\} \\
 & | & \neg \textit{cond} & \\
 & | & \textit{cond} \diamond \textit{cond} & \diamond \in \{\wedge, \vee\}
\end{array}
$$

Statements

$$
\begin{array}{lll}
\textit{stat} & ::= & V \leftarrow \textit{expr} \\
 & | & \textbf{if } \textit{cond} \textbf{ then } \textit{stat} \textbf{ else } \textit{stat} \\
 & | & \textbf{while } \textit{cond} \textbf{ do } \textit{stat} \\
 & | & \textit{stat};\ \textit{stat} \\
 & | & \textbf{skip}
\end{array}
$$

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Concrete semantics

Classic non-deterministic concrete semantics, in denotational style:

$\mathbb{E}[\![\, expr \,]\!] : \mathcal{E} \to \mathcal{P}(\mathbb{Z})$ *(arithmetic expressions)*

$$
\begin{array}{ll}
\mathbb{E}[\![\, V \,]\!]\rho & \stackrel{\text{def}}{=} \{\rho(V)\} \\
\mathbb{E}[\![\, c \,]\!]\rho & \stackrel{\text{def}}{=} \{c\} \\
\mathbb{E}[\![\, \textbf{rand}(a,b) \,]\!]\rho & \stackrel{\text{def}}{=} \{\, x \mid a \leq x \leq b \,\} \\
\mathbb{E}[\![\, -e \,]\!]\rho & \stackrel{\text{def}}{=} \{\, -v \mid v \in \mathbb{E}[\![\, e \,]\!]\rho \,\} \\
\mathbb{E}[\![\, e_1 \diamond e_2 \,]\!]\rho & \stackrel{\text{def}}{=} \{\, v_1 \diamond v_2 \mid v_1 \in \mathbb{E}[\![\, e_1 \,]\!]\rho,\ v_2 \in \mathbb{E}[\![\, e_2 \,]\!]\rho,\ \diamond \neq / \vee v_2 \neq 0 \,\}
\end{array}
$$

$\mathbb{C}[\![\, cond \,]\!] : \mathcal{E} \to \mathcal{P}(\{\text{true}, \text{false}\})$ *(boolean conditions)*

$$
\begin{array}{ll}
\mathbb{C}[\![\, \neg c \,]\!]\rho & \stackrel{\text{def}}{=} \{\, \neg v \mid v \in \mathbb{C}[\![\, c \,]\!]\rho \,\} \\
\mathbb{C}[\![\, c_1 \diamond c_2 \,]\!]\rho & \stackrel{\text{def}}{=} \{\, v_1 \diamond v_2 \mid v_1 \in \mathbb{C}[\![\, c_1 \,]\!]\rho,\ v_2 \in \mathbb{C}[\![\, c_2 \,]\!]\rho \,\} \\
\mathbb{C}[\![\, e_1 \bowtie e_2 \,]\!]\rho & \stackrel{\text{def}}{=} \{\, \text{true} \mid \exists v_1 \in \mathbb{E}[\![\, e_1 \,]\!]\rho,\ v_2 \in \mathbb{E}[\![\, e_2 \,]\!]\rho \colon v_1 \bowtie v_2 \,\} \cup \\
 & \quad\ \{\, \text{false} \mid \exists v_1 \in \mathbb{E}[\![\, e_1 \,]\!]\rho,\ v_2 \in \mathbb{E}[\![\, e_2 \,]\!]\rho \colon v_1 \not\bowtie v_2 \,\}
\end{array}
$$

where $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{V} \to \mathbb{Z}$

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Concrete semantics

$\mathbb{S}[\![\, stat \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$

$$
\begin{array}{ll}
\mathbb{S}[\![\, \textbf{skip} \,]\!]R & \stackrel{\text{def}}{=} R \\
\mathbb{S}[\![\, s_1;\ s_2 \,]\!]R & \stackrel{\text{def}}{=} \mathbb{S}[\![\, s_2 \,]\!](\mathbb{S}[\![\, s_1 \,]\!]R) \\
\mathbb{S}[\![\, V \leftarrow e \,]\!]R & \stackrel{\text{def}}{=} \{\, \rho[V \mapsto v] \mid \rho \in R,\ v \in \mathbb{E}[\![\, e \,]\!]\rho \,\} \\
\mathbb{S}[\![\, \textbf{if}\ c\ \textbf{then}\ s_1\ \textbf{else}\ s_2 \,]\!]R & \stackrel{\text{def}}{=} \mathbb{S}[\![\, s_1 \,]\!](\mathbb{S}[\![\, c? \,]\!]R) \cup \mathbb{S}[\![\, s_2 \,]\!](\mathbb{S}[\![\, \neg c? \,]\!]R) \\
\mathbb{S}[\![\, \textbf{while}\ c\ \textbf{do}\ s \,]\!]R & \stackrel{\text{def}}{=} \mathbb{S}[\![\, \neg c? \,]\!](\text{lfp}\ \lambda I.\, R \cup \mathbb{S}[\![\, s \,]\!](\mathbb{S}[\![\, c? \,]\!]I))
\end{array}
$$

where

$$
\mathbb{S}[\![\, c? \,]\!]R \stackrel{\text{def}}{=} \{\, \rho \in R \mid \text{true} \in \mathbb{C}[\![\, c \,]\!]\rho \,\}
$$

$\mathbb{S}[\![\, stat \,]\!]$ is a $\cup-$morphism in the complete lattice $(\mathcal{P}(\mathcal{E}), \subseteq, \cup, \cap, \emptyset, \mathcal{E})$

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Reminder: Non-relational abstractions

Reminder: we compose two abstractions:

- $\mathcal{P}(\mathcal{V} \to \mathbb{Z})$ is abstracted as $\mathcal{V} \to \mathcal{P}(\mathbb{Z})$ (forget relationship)
- $\mathcal{P}(\mathbb{Z})$ is abstracted as intervals $\mathcal{I}$ (keep only bounds)

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Interval expression evaluation

$\mathbb{E}^\sharp[\![\, expr \,]\!] : \mathcal{E}^\sharp \to \mathcal{I}$
interval version of $\mathbb{E}[\![\, expr \,]\!] : \mathcal{E} \to \mathcal{P}(\mathbb{Z})$

Definition by structural induction, very similar to $\mathbb{E}[\![\, expr \,]\!]$

$$
\begin{array}{ll}
\mathbb{E}^\sharp[\![\, V \,]\!]X^\sharp & \stackrel{\text{def}}{=} X^\sharp(V) \\
\mathbb{E}^\sharp[\![\, c \,]\!]X^\sharp & \stackrel{\text{def}}{=} [c, c] \\
\mathbb{E}^\sharp[\![\, \textbf{rand}(a, b) \,]\!]X^\sharp & \stackrel{\text{def}}{=} [a, b] \\
\mathbb{E}^\sharp[\![\, -e \,]\!]X^\sharp & \stackrel{\text{def}}{=} -^\sharp\ \mathbb{E}^\sharp[\![\, e \,]\!]X^\sharp \\
\mathbb{E}^\sharp[\![\, e_1 \diamond e_2 \,]\!]X^\sharp & \stackrel{\text{def}}{=} \mathbb{E}^\sharp[\![\, e_1 \,]\!]X^\sharp\ \diamond^\sharp\ \mathbb{E}^\sharp[\![\, e_2 \,]\!]X^\sharp
\end{array}
$$

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Interval arithmetic

- $-^\sharp\ [a,b] = [-b, -a]$
- $[a,b]\ +^\sharp\ [c,d] = [a+c, b+d]$
- $[a,b]\ -^\sharp\ [c,d] = [a-d, b-c]$
- $\forall i \in \mathcal{I}$: $-^\sharp\ \bot = \bot\ +^\sharp\ i = i\ +^\sharp\ \bot = \cdots = \bot$ (strictness)

where: $+$ and $-$ is extended to $+\infty$, $-\infty$ as:
$\forall x \in \mathbb{Z}$: $(+\infty) + x = +\infty$, $(-\infty) + x = -\infty$, $-(+\infty) = (-\infty), \ldots$

- $[a,b]\ \times^\sharp\ [c,d] = [\ \min(a \times c,\ a \times d,\ b \times c,\ b \times d),$
  $\max(a \times c,\ a \times d,\ b \times c,\ b \times d)\ ]$

where $\times$ is extended to $+\infty$ and $-\infty$ by the rule of signs:
$c \times (+\infty) = (+\infty)$ if $c > 0$, $(-\infty)$ if $c < 0$
$c \times (-\infty) = (-\infty)$ if $c > 0$, $(+\infty)$ if $c < 0$

we also need the non-standard rule: $0 \times (+\infty) = 0 \times (-\infty) = 0$

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Summary of the abstract semantics

$$\mathbb{S}^\sharp[\![\, \textbf{skip} \,]\!]X^\sharp \stackrel{\text{def}}{=} X^\sharp$$

$$\mathbb{S}^\sharp[\![\, s_1; s_2 \,]\!]X^\sharp \stackrel{\text{def}}{=} \mathbb{S}^\sharp[\![\, s_2 \,]\!](\mathbb{S}^\sharp[\![\, s_1 \,]\!]X^\sharp)$$

$$\mathbb{S}^\sharp[\![\, V \leftarrow e \,]\!]X^\sharp \stackrel{\text{def}}{=} \begin{cases} X^\sharp[V \mapsto \mathbb{E}^\sharp[\![\, e \,]\!]X^\sharp] & \text{if } \mathbb{E}^\sharp[\![\, e \,]\!]X^\sharp \neq \bot \\ \dot{\bot} & \text{if } \mathbb{E}^\sharp[\![\, e \,]\!]X^\sharp = \bot \end{cases}$$

$$\mathbb{S}^\sharp[\![\, \textbf{if } c \textbf{ then } s_1 \textbf{ else } s_2 \,]\!]X^\sharp \stackrel{\text{def}}{=} \mathbb{S}^\sharp[\![\, s_1 \,]\!](\mathbb{S}^\sharp[\![\, c? \,]\!]X^\sharp) \mathbin{\dot\cup^\sharp} \mathbb{S}^\sharp[\![\, s_2 \,]\!](\mathbb{S}^\sharp[\![\, \neg c? \,]\!]X^\sharp)$$

$$\mathbb{S}^\sharp[\![\, \textbf{while } c \textbf{ do } s \,]\!]X^\sharp \stackrel{\text{def}}{=} \mathbb{S}^\sharp[\![\, \neg c? \,]\!](\lim \lambda I^\sharp.\, I^\sharp \mathbin{\dot\nabla} (X^\sharp \mathbin{\dot\cup^\sharp} \mathbb{S}^\sharp[\![\, s \,]\!](\mathbb{S}^\sharp[\![\, c? \,]\!]I^\sharp)))$$

(next slides: extending the language with assertions and local variables)

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Convergence acceleration

Widening: binary operator $\triangledown : \mathcal{E}^\sharp \times \mathcal{E}^\sharp \to \mathcal{E}^\sharp$ such that:

- $\gamma(X^\sharp) \cup \gamma(Y^\sharp) \subseteq \gamma(X^\sharp \triangledown Y^\sharp)$ (sound abstraction of $\cup$)
- for any sequence $(X_n^\sharp)_{n\in\mathbb{N}}$, the sequence $(Y_n^\sharp)_{n\in\mathbb{N}}$

$$\begin{cases} Y_0^\sharp & \stackrel{\text{def}}{=} X_0^\sharp \\ Y_{n+1}^\sharp & \stackrel{\text{def}}{=} Y_n^\sharp \triangledown X_{n+1}^\sharp \end{cases}$$

stabilizes in finite time: $\exists N \in \mathbb{N}$: $Y_N^\sharp = Y_{N+1}^\sharp$

Fixpoint approximation theorem:

- the sequence $X_{n+1}^\sharp \stackrel{\text{def}}{=} X_n^\sharp \triangledown F^\sharp(X_n^\sharp)$ stabilizes in finite time
- when $X_{N+1}^\sharp \sqsubseteq X_N^\sharp$, then $X_N^\sharp$ abstracts $\mathrm{lfp}\, F$

Soundness proof: assume $X_{N+1}^\sharp \sqsubseteq X_N^\sharp$, then

$\gamma(X_N^\sharp) \supseteq \gamma(X_{N+1}^\sharp) = \gamma(X_N^\sharp \triangledown F^\sharp(X_N^\sharp)) \supseteq \gamma(F^\sharp(X_N^\sharp)) \supseteq F(\gamma(X_N^\sharp))$

$\gamma(X_N^\sharp)$ is a post-fixpoint of F, but $\mathrm{lfp}\, F$ is F's least post-fixpoint, so, $\gamma(X_N^\sharp) \supseteq \mathrm{lfp}\, F$

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Interval widening

Interval widening $\triangledown : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$

$$\forall I \in \mathcal{I}: \bot \triangledown I = I \triangledown \bot = I$$

$$[a,b] \triangledown [c,d] \stackrel{\text{def}}{=} \left[\begin{cases} a & \text{if } a \leq c \\ -\infty & \text{if } a > c \end{cases}, \begin{cases} b & \text{if } b \geq d \\ +\infty & \text{if } b < d \end{cases} \right]$$

- an unstable lower bound is put to $-\infty$
- an unstable upper bound is put to $+\infty$
- once at $-\infty$ or $+\infty$, the bound becomes stable

Point-wise lifting: $\dot{\triangledown} : \mathcal{E}^\sharp \times \mathcal{E}^\sharp \to \mathcal{E}^\sharp$

$$X^\sharp \mathbin{\dot{\triangledown}} Y^\sharp \stackrel{\text{def}}{=} \lambda V \in \mathcal{V}. X^\sharp(V) \triangledown Y^\sharp(V)$$

extrapolate each variable independently

$\Longrightarrow$ stabilization in at most $2|\mathcal{V}|$ iterations

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2 Operational semantics

3 Denotational semantics

4 Axiomatic semantics

5 Abstract interpretation
- Principle
- Example of abstract interpretation
- Interval analysis, more formally
- Alarms
- Abstract domains

6 Industrial state-of-the-use at Airbus

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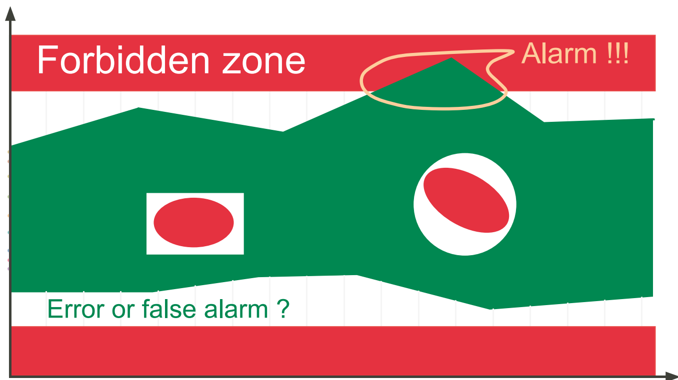
Alarms



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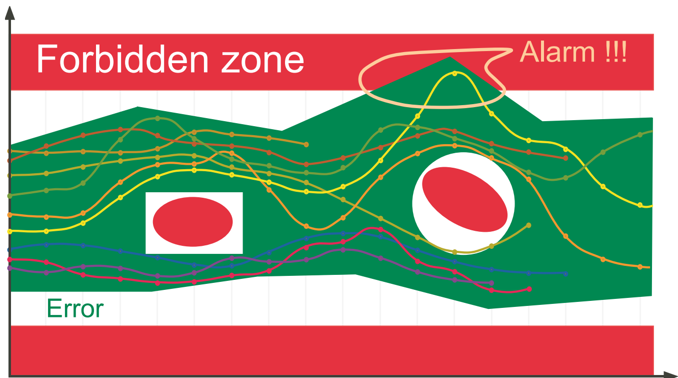
True error



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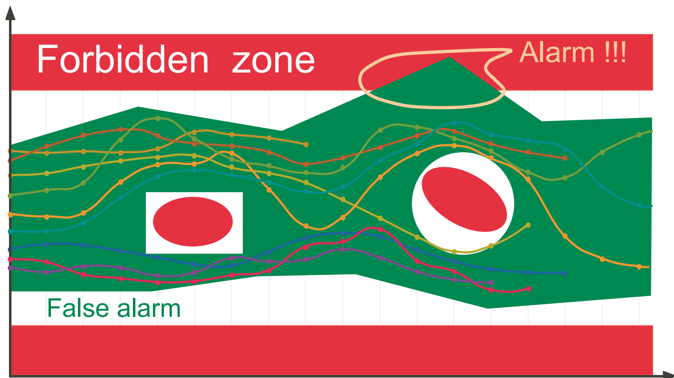
Incompleteness ⇒ false alarms



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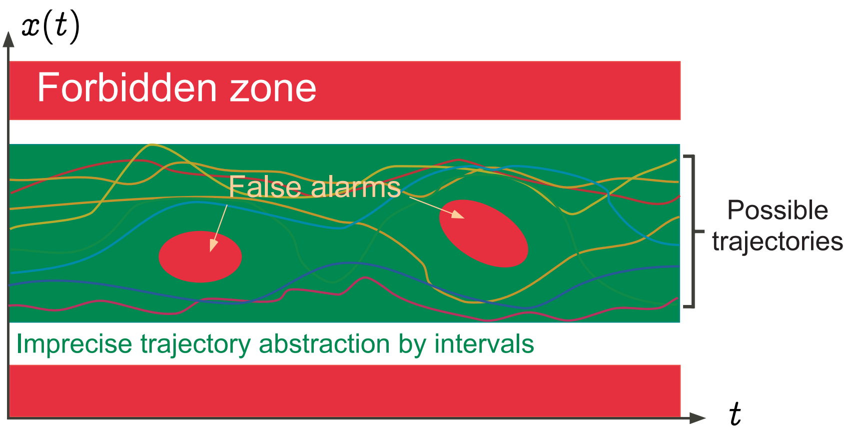
False alarms (e.g. interval analysis)



$\Rightarrow$ need for abstraction refinement

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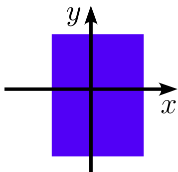
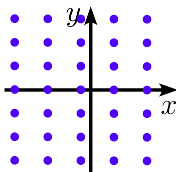
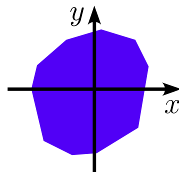
Examples of numerical abstract domains



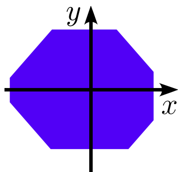
intervals
$x, y \in [a, b]$



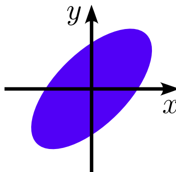
congruences
$x, y \in a\mathbb{Z} + b$



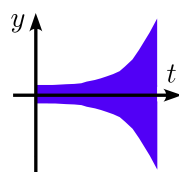
polyhedra
$\bigwedge \sum_i a_i x_i \leq b$



octagons
$\bigwedge \pm x \pm y \leq c$



ellipsoids
$x^2 + by^2 - axy \leq d$



geometric deviations
$|y| \leq a(1+b)^{kt}$

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Combinations of abstractions

Semantics

(In)finite sets of (in)finite traces

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Combinations of abstractions

Invariant abstraction: set of reachable states

Set of points (x_i, y_i), Hoare logic

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Combinations of abstractions

Sign abstraction

y

x

$x \geq 0,\ y \geq 0$

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Combinations of abstractions

Interval abstraction

y

x

$a \leq x \leq b,\ c \leq y \leq d$

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Combinations of abstractions

Octagon abstraction

y

x

$\pm x \pm y \leq b$

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Combinations of abstractions

Polyhedral abstraction

y

x

$$a \cdot x + b \cdot y \leq c$$

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Combinations of abstractions

Ellipsoidal abstraction

y

x

$$(x-a)^2 + (y-b)^2 \leq c$$

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Combinations of abstractions

Exponential abstraction

$$a^x \leq y$$

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- Static analysis of source code in today's industrial processes
- Focus on run-time error analysis
  - Today: Astrée
  - Soon: AstréeA extension
- Ongoing technology transfers
- Static analysis of executables in today's industrial processes

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Proof of absence of run-time errors for fly-by-wire

- AI-based static analysis of **C** source code
- ASTRÉE by AbsInt (CNRS/ENS license)
- DAL A A340/A380/A400M control software up to 650 Kloc (A350 soon)

Unit Proof on DAL A software subsets

- WP-based program proof at **C** function level
- deployed on A380/A400M/A350 fly-by-wire subsets
- Caveat (CEA) + Alt-Ergo SMT-solver (INRIA) qualified wrt. DO-178B

Data & control flow analyses

- AI-based static analysis of **C** code
- local analyses (small subsets of the call graph)
- Fan-C (Airbus), a Frama-C (CEA-INRIA) plugin

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Static analysis of synchronous C programs

Concrete semantics and specification

Concrete semantics source

- C99 standard (portable **C** programs)
- IEEE 754-1985 standard (floating-point arithmetic)
- architecture parameters (`sizeof`, endianess, `struct`, etc.)
- compiler and linker parameters (initialization, etc.)

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Static analysis of synchronous C programs

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Concrete semantics source

- C99 standard (portable **C** programs)
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- architecture parameters (`sizeof`, endianess, `struct`, etc.)
- compiler and linker parameters (initialization, etc.)

Run-time errors

- overflows in float, integer, `enum` arithmetic and cast
- division, modulo by 0 on integers and floats
- invalid pointer arithmetic or dereferencing
- violation of user-specified assertions

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Some of ASTRÉE's numerical abstract domains

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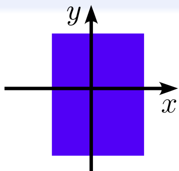
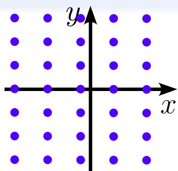


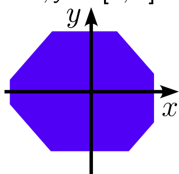
intervals

$x, y \in [a, b]$



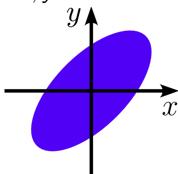
congruences

$x, y \in a\mathbb{Z} + b$



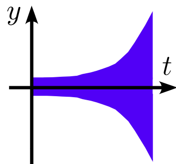
octagons

$\bigwedge \pm x \pm y \leq c$



ellipsoids

$x^2 + by^2 - axy \leq d$



geometric deviations

$|y| \leq a(1 + b)^{kt}$

relational domains are necessary to infer precise bounds

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The ASTRÉE static analyser

Analyseur Statique de logiciels Temps-RÉel Embarqués

A static analyzer for C programs

- developed by CNRS/ENS (from 2002) and AbsInt GmbH
- commercialised by AbsInt since 2010

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Characteristics

1. sound and automatic, scales up to very large programs
2. specialised for control-command programs $\Rightarrow$ few false alarms
3. parametric $\Rightarrow$ fine-tuning by end-users

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Results at Airbus

automatic proof of absence of *run-time error*
on FBW control programs of LR, A380 and A400M

≃ 8 hours for 650 000 lines of **C**

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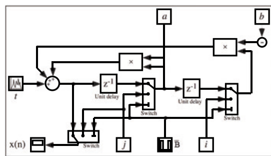
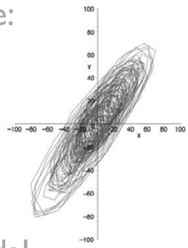
Domain-specific abstraction : ellipsoids

An abstraction for invariants of digital filters

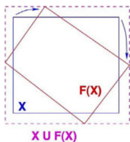
2nd order filter:



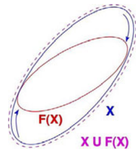
Execution trace:



Unstable polyhedral abstraction:



Stable ellipsoidal abstraction:



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Static analysis of parallel **C** programs

Concrete semantics and specification

Model: real-time operating system

- fixed set of concurrent threads on a single processor
- shared memory (implicit communications)
- synchronisation primitives (fixed set of mutexes)
- real-time scheduling with fixed priorities (priority-based)

  e.g. A380/A400M/A350 IMA, A350 ASF, SA/LR ATSU
- mono-threaded startup $\neq$ multi-threaded run (restriction)

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  e.g. A380/A400M/A350 IMA, A350 ASF, SA/LR ATSU
- mono-threaded startup $\neq$ multi-threaded run (restriction)

Run-time errors

- classic C run-time errors (overflows, invalid pointers, etc.)
- unprotected data-races (report & factor in the analysis)
- incorrect system calls
- but NOT deadlocks, livelocks, priority inversions

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The ASTRÉEA prototype static analyser

An extension of ASTRÉE

- Analyseur Statique de logiciels Temps-RÉel Embarqués Asynchrones
- developed by CNRS/ENS/INRIA since 2009

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An extension of ASTRÉE

- Analyseur Statique de logiciels Temps-RÉel Embarqués Asynchrones
- developed by CNRS/ENS/INRIA since 2009

On-going work with Airbus

static analysis of A380 FWS

- 15 processes
- $\simeq$ 2 million lines of **C**
- nested loops, complex data structures

$\simeq$ 1000 alarms

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Numerical accuracy assessment for fly-by-wire and applications

- AI-based static analysis of **C** source code
- FLUCTUAT (CEA): successful experiments on subsets of control programs
- first target: basic numerical operators
- automate (manual) accuracy analyses

Certified compilation

- CompCert (INRIA) certified compiler
  formally verified semantic equivalence
- complementary to formal verification of **C** code
- optimizations improve WCET...
  ...while maintaining sufficient tracability
- may help save on COM/MON dissymmetry?
- requires well-defined semantics at **C** level
  $\Rightarrow$ *run-time error* analysis

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  - Today: Astrée
  - Soon: AstréeA extension
- Ongoing technology transfers
- Static analysis of executables in today's industrial processes

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AbsInt binary static analyzers used for certification

WCET Analysis on DAL A time-critical software products

- AI-based static analysis of executable code
- deployed on A340/A380/A400M/A350
  e.g. PowerPC MPC755 and 7448, TI TMS320C33
  sequential/synchronous programs, up to 650 klo**C**
- aiT WCET qualified wrt. DO-178B

Stack Analysis on all embedded software products

- AI-based static analysis of executable code
- deployed on A330/A340/A380/A400M/A350
  e.g. x86, PowerPC 755, 7448, 8610, TI TMS320C3x
  also multithreaded asynchronous programs with complex data structures up to 2 Mlo**C**
- StackAnalyzer qualified wrt. DO-178B

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Conclusion

Domaines des mathématiques dont nous avons besoin:

- Théorie des ensembles
- Ordres, théorie des treillis, points fixes
- Logique formelle, décidabilité, calculabilité
- Théorie de la preuve, théorie des types

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Outline

7 Orderings, lattices, fixpoints
- Basic definitions on orderings
- Operators over a poset and fixpoints

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Order relations

Very useful in semantics and verification:

- **logical ordering**, expresses **implication** of logical facts
- **computational ordering**, useful to establish well-foundedness of fixpoint definitions, and for termination

Definition: partially ordered set (poset)

Let a set $\mathcal{S}$ and a binary relation $\sqsubseteq \subseteq \mathcal{S} \times \mathcal{S}$ over $\mathcal{S}$. Then, $\sqsubseteq$ is an **order relation** (and $(\mathcal{S}, \sqsubseteq)$ is called a **poset**) if and only if it is

- **reflexive:** $\forall x \in \mathcal{S},\ x \sqsubseteq x$
- **transitive:** $\forall x, y, z \in \mathcal{S},\ x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z$
- **antisymmetric:** $\forall x, y \in \mathcal{S},\ x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y$

- notation: $x \sqsubset y ::= (x \sqsubseteq y \wedge x \neq y)$

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Graphical representation

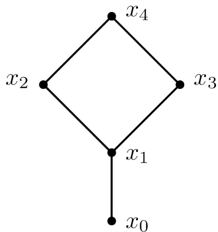
We often use **Hasse diagrams** to represent posets:

Extensive definition:

- $\mathcal{S} = \{x_0, x_1, x_2, x_3, x_4\}$
- $\sqsubset$ defined by:

$$
\begin{aligned}
x_0 &\sqsubset x_1 \\
x_1 &\sqsubset x_2 \\
x_1 &\sqsubset x_3 \\
x_2 &\sqsubset x_4 \\
x_3 &\sqsubset x_4
\end{aligned}
$$

Diagram:



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Total ordering

Definition: total order relation

Order relation $\sqsubseteq$ over $\mathcal{S}$ is a **total** order if and only if

$$\forall x, y \in \mathcal{S},\ x \sqsubseteq y \vee y \sqsubseteq x$$

- $(\mathbb{R}, \leq)$ is a total ordering
- if set $\mathcal{S}$ has at least two distinct elements x, y then its powerset $(\mathcal{P}(\mathcal{S}), \subseteq)$ is **not** a total order
  indeed $\{x\}, \{y\}$ cannot be compared
- most of the order relations we will use are *not* be total

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Minimum and maximum elements

Definition: extremal elements

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $\mathcal{S}' \subseteq \mathcal{S}$. Then x is

- **minimum element** of $\mathcal{S}'$ if and only if
  $x \in \mathcal{S}' \land \forall y \in \mathcal{S}',\ x \sqsubseteq y$
- **maximum element** of $\mathcal{S}'$ if and only if
  $x \in \mathcal{S}' \land \forall y \in \mathcal{S}',\ y \sqsubseteq x$

- maximum and minimum elements **may not exist**
  example: $\{\{x\}, \{y\}\}$ in the powerset, where $x \neq y$
- **infimum $\perp$ ("bottom"):** minimum element of $\mathcal{S}$
- **supremum $\top$ ("top"):** maximum element of $\mathcal{S}$

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Upper bounds and least upper bound

Definition: bounds

Given poset $(\mathcal{S}, \sqsubseteq)$ and $\mathcal{S}' \subseteq \mathcal{S}$, then $x \in \mathcal{S}$ is

- an **upper bound** of $\mathcal{S}'$ if

$$\forall y \in \mathcal{S}',\ y \sqsubseteq x$$

- the **least upper bound** (lub) of $\mathcal{S}'$ (noted $\sqcup \mathcal{S}'$) if

$$\forall y \in \mathcal{S}',\ y \sqsubseteq x \wedge \forall z \in \mathcal{S}, (\forall y \in \mathcal{S}',\ y \sqsubseteq z) \implies x \sqsubseteq z$$

- if it exists, the least upper bound is **unique**:
  if x, y are least upper bounds of $\mathcal{S}$, then $x \sqsubseteq y$ and $y \sqsubseteq x$,
  thus $x = y$ by antisymmetry
- notation: $x \sqcup y ::= \sqcup\{x, y\}$
- upper bounds and least upper bounds **may not exist**
- **dual notions:** lower bound, greatest lower bound (glb, noted $\sqcap \mathcal{S}'$)

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Duality principle

So far all definitions admit a symmetric counterpart

- given an order relation $\sqsubseteq$, $\mathcal{R}$ defined by $x \mathcal{R} y \iff y \sqsubseteq x$ is also an order relation
- thus all properties that can be proved about $\sqsubseteq$ also have a symmetric property that also holds

This is the **duality principle**:

minimum element	maximum element
infimum	supremum
lower bound	upper bound
greatest lower bound	least upper bound

... more to follow

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Complete lattice

Definition: complete lattice

A **complete lattice** is a tuple $(\mathcal{S}, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ where:

- $(\mathcal{S}, \sqsubseteq)$ is a poset
- $\bot$ is the infimum of $\mathcal{S}$
- $\top$ is the supremum of $\mathcal{S}$
- any subset $\mathcal{S}'$ of $\mathcal{S}$ has a lub $\sqcup \mathcal{S}'$ and a glb $\sqcap \mathcal{S}'$

Properties:

- $\bot = \sqcup \emptyset = \sqcap \mathcal{S}$
- $\top = \sqcap \emptyset = \sqcup \mathcal{S}$

Example: the **powerset** $(\mathcal{P}(\mathcal{S}), \subseteq, \emptyset, \mathcal{S}, \cup, \cap)$ of set $\mathcal{S}$ is a complete lattice

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Lattice

The existence of lubs and glbs for all subsets is often a very strong property, that may not be met:

Definition: lattice

A **lattice** is a tuple $(\mathcal{S}, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ where:

- $(\mathcal{S}, \sqsubseteq)$ is a poset
- $\bot$ is the infimum of $\mathcal{S}$
- $\top$ is the supremum of $\mathcal{S}$
- any **pair** $\{x, y\}$ of $\mathcal{S}$ has a lub $x \sqcup y$ and a glb $x \sqcap y$

- let $\mathcal{Q} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$;
  then $(\mathcal{Q}, \leq)$ is a **lattice** but **not a complete lattice**
  indeed, $\{q \in \mathcal{Q} \mid q \leq \frac{\sqrt{2}}{2}\}$ has no lub in $\mathcal{Q}$
- property: a **finite** lattice is also a complete lattice

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Chains

Definition: increasing chain

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $\mathcal{C} \subseteq \mathcal{S}$.
It is an **increasing chain** if and only if

- it has an infimum
- poset $(\mathcal{C}, \sqsubseteq)$ is total (i.e., any two elements can be compared)

Example, in the powerset $(\mathcal{P}(\mathbb{N}), \subseteq)$:

$$\mathcal{C} = \{c_i \mid i \in \mathbb{N}\} \quad \text{where} \quad c_i = \{2^0, 2^2, \ldots, 2^i\}$$

Definition: increasing chain condition

Poset $(\mathcal{S}, \sqsubseteq)$ **satisfies the increasing chain condition** if and only if any increasing chain $\mathcal{C} \subseteq \mathcal{S}$ is finite.

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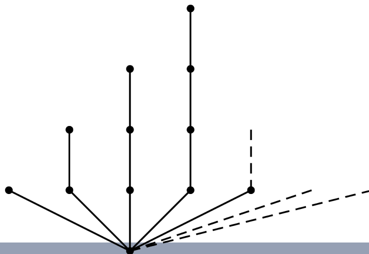
Complete partial orders

Definition: complete partial order

A **complete partial order** (cpo) is a poset $(\mathcal{S}, \sqsubseteq)$ such that any increasing chain $\mathcal{C}$ of $\mathcal{S}$ has a least upper bound. A **pointed cpo** is a cpo with an infimum $\perp$.

- clearly, any complete lattice is a cpo
- the opposite is not true:



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Operators over a poset

Definition: operators and orderings

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $\phi : \mathcal{S} \to \mathcal{S}$ be an operator over $\mathcal{S}$.
Then, ϕ is:

- **monotone** if and only if $\forall x, y \in \mathcal{S},\ x \sqsubseteq y \Longrightarrow \phi(x) \sqsubseteq \phi(y)$
- **continuous** if and only if, for any chain $\mathcal{S}' \subseteq \mathcal{S}$ then:
  $$\begin{cases} \text{if } \sqcup \mathcal{S}' \text{ exists, so does } \sqcup \{\phi(x) \mid x \in \mathcal{S}'\} \\ \text{and } \phi(\sqcup \mathcal{S}') = \sqcup\{\phi(x) \mid x \in \mathcal{S}'\} \end{cases}$$
- **$\sqcup$-preserving** if and only if:
  $$\forall \mathcal{S}' \subseteq \mathcal{S}, \begin{cases} \text{if } \sqcup \mathcal{S}' \text{ exists, then } \sqcup \{\phi(x) \mid x \in \mathcal{S}'\} \text{ exists} \\ \text{and } \phi(\sqcup \mathcal{S}') = \sqcup\{\phi(x) \mid x \in \mathcal{S}'\} \end{cases}$$

Notes:

- "monotone" in English means "*croissante*" in French ; "*décroissante*" translates into "anti-monotone" and "monotone" into "*isotone*"

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Operators over a poset

A few interesting properties:

- **continuous $\Rightarrow$ monotone:**
  if ϕ is monotone, and $x, y \in \mathcal{S}$ are such that $x \sqsubseteq y$, then $\{x, y\}$ is a chain with lub y, thus $\phi(x) \sqcup \phi(y)$ exists and is equal to $\phi(\sqcup\{x, y\}) = \phi(y)$; therefore $\phi(x) \sqsubseteq \phi(y)$.
- **$\sqcup$-preserving $\Rightarrow$ monotone:**
  same argument.

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Fixpoints

Definition: fixpoints

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $f : \mathcal{S} \to \mathcal{S}$ be an operator over $\mathcal{S}$.

- a **fixpoint** of ϕ is an element x such that $\phi(x) = x$
- a **pre-fixpoint** of ϕ is an element x such that $x \sqsubseteq \phi(x)$
- a **post-fixpoint** of ϕ is an element x such that $\phi(x) \sqsubseteq x$
- the **least fixpoint** $\operatorname{lfp} \phi$ of ϕ (if it exists, it is unique) is the smallest fixpoint of ϕ
- the **greatest fixpoint** $\operatorname{gfp} \phi$ of ϕ (if it exists, it is unique) is the greatest fixpoint of ϕ

Note: the existence of a least fixpoint, a greatest fixpoint or even a fixpoint is *not guaranteed*; we will see several theorems that establish their existence under specific assumptions...

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Tarski's Theorem

Theorem

Let $(\mathcal{S}, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ be a complete lattice and $\phi : \mathcal{S} \to \mathcal{S}$ be a monotone operator over $\mathcal{S}$. Then:

1. ϕ has a least fixpoint $\text{lfp}\,\phi$ and $\text{lfp}\,\phi = \sqcap\{x \in \mathcal{S} \mid \phi(x) \sqsubseteq x\}$.
2. ϕ has a greatest fixpoint $\text{gfp}\,\phi$ and $\text{gfp}\,\phi = \sqcup\{x \in \mathcal{S} \mid x \sqsubseteq \phi(x)\}$.
3. the set of fixpoints of ϕ is a complete lattice.

Proof of point 1:

We let $X = \{x \in \mathcal{S} \mid \phi(x) \sqsubseteq x\}$ and $x_0 = \sqcap X$.

Let $y \in X$:

- $x_0 \sqsubseteq y$ by definition of the glb;
- thus, since ϕ is monotone, $\phi(x_0) \sqsubseteq \phi(y)$;
- thus, $\phi(x_0) \sqsubseteq y$ since $\phi(y) \sqsubseteq y$, by definition of X.

Therefore $\phi(x_0) \sqsubseteq x_0$, since $x_0 = \sqcap X$.

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Tarski's Theorem

We proved that $\phi(x_0) \sqsubseteq x_0$. We derive from this that:

- $\phi(\phi(x_0)) \sqsubseteq \phi(x_0)$ since ϕ is monotone;
- $\phi(x_0)$ is a post-fixpoint of ϕ, thus $\phi(x_0) \in X$;
- $x_0 \sqsubseteq \phi(x_0)$ by definition of the greatest lower bound

We have established both inclusions so $\phi(x_0) = x_0$.

Proof of point 2: similar, by duality.

Proof of point 3:

- if X is a set of fixpoints of ϕ, we need to consider ϕ over $\{y \in \mathcal{S} \mid y \sqsubseteq_{\mathcal{S}} \sqcap X\}$ to establish the existence of a **glb of X in the poset of fixpoints**
- the existence of **least upper bounds in the poset of fixpoints** follows by duality

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Kleene's Theorem

Tarski's theorem guarantees existence of an lfp, but is not constructive.

Theorem

Let $(\mathcal{S}, \sqsubseteq, \bot)$ be a pointed cpo and $\phi : \mathcal{S} \rightarrow \mathcal{S}$ be a continuous operator over $\mathcal{S}$. Then ϕ has a least fixpoint, and

$$\mathrm{lfp}\,\phi = \bigsqcup_{n \in \mathbb{N}} \phi^n(\bot)$$

First, we prove **the existence of the lub**:

Since ϕ is continuous, it is also monotone. We can prove by induction over n that $\{\phi^n(\bot) \mid n \in \mathbb{N}\}$ is a chain:

- $\phi^0(\bot) = \bot \sqsubseteq \phi(\bot)$ by definition of the infimum;
- if $\phi^n(\bot) \sqsubseteq \phi^{n+1}(\bot)$, then $\phi^{n+1}(\bot) = \phi(\phi^n(\bot)) \sqsubseteq \phi(\phi^{n+1}(\bot)) = \phi^{n+2}(\bot)$

By definition of the cpo structure, the lub exists. We let x_0 denote it.

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Kleene's Theorem

Secondly, we prove that **it is a fixpoint of ϕ**:

Since ϕ is continuous, $\{\phi^{n+1}(\bot) \mid n \in \mathbb{N}\}$ has a lub, and

$$\begin{array}{rcll} \phi(x_0) & = & \phi(\sqcup\{\phi^n(\bot) \mid n \in \mathbb{N}\}) & \\ & = & \{\sqcup\phi^{n+1}(\bot) \mid n \in \mathbb{N}\} & \text{by continuity of } \phi \\ & = & \bot \sqcup \{\sqcup\phi^{n+1}(\bot) \mid n \in \mathbb{N}\} & \text{by definition of } \bot \\ & = & x_0 & \text{by simple rewrite} \end{array}$$

Last, we show that it is the **least** fixpoint:

Let x_1 denote another fixpoint of ϕ. We show by induction over n that $\phi^n(\bot) \sqsubseteq x_1$:

- $\phi^0(\bot) = \bot \sqsubseteq x_1$ by definition of $\bot$;
- if $\phi^n(\bot) \sqsubseteq x_1$, then $\phi^{n+1}(\bot) \sqsubseteq \phi(x_1) = x_1$ by monotony, and since x_1 is a fixpoint.

By definition of the lub, $x_0 \sqsubseteq x_1$

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Automata example, constructive

We can now state a **constructive definition** of the automaton semantics. Operator ϕ is defined by

$$\phi(f) = \lambda(q \in Q) \cdot \begin{cases} \{\epsilon\} \cup \phi_0(f)(q_\mathrm{i}) & \text{if } q = q_\mathrm{i} \\ \phi_0(f)(q) & \text{otherwise} \end{cases}$$

Proof steps:

- ϕ is continuous
- thus, Kleene's theorem applies so $\operatorname{lfp} \phi$ exists and $\operatorname{lfp} \phi = \bigcup_{n \in \mathbb{N}} \phi^n(\bot)$...
  ... this actually saves the double inclusion proof to establish that $[\![\mathcal{A}]\!] = \operatorname{lfp} \phi$

Furthermore, $[\![\mathcal{A}]\!] = \bigcup_{n \in \mathbb{N}} \phi^n(\bot)$.

This fixpoint definition will be very useful to infer or verify semantic properties.

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Duality principle

We can extend the duality notion:

monotone	monotone
anti-monotone	anti-monotone
post-fixpoint	pre-fixpoint
least fixpoint	greatest fixpoint
increasing chain	decreasing chain

Furthermore both Tarski's theorem and Kleene's theorem have a dual version (Tarski's theorem mostly encloses its own dual, except for the definition of the gfp).

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Questions?

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