Journées APMEP, 10 Octobre 2015, LAON

Identification de Sources dans les Modèles de Transport d'Oxygène. (Pollution Organique).

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Pollution Organique

Bassin Versant Loire-Bretagne

Origine de la Pollution Organique

- Agriculture et Elevage : 60%
- Domestique : 30%
- Industrielle : 10%

Source (2011): Une Etude relativement récente qui traîne sur l'un des sites WEB avec la terminaison .gouv.fr

Eutrophisation (I)

Excès de matière organique $\Longrightarrow$ Excès de nutriments $\Longrightarrow$ Eutrophisation

Le Thouet, en 2009, colonisé par l'elodée dense.

Cette rivière était saine en 2007.

(Source : WEB!).

Eutrophisation (II)

Lac Erié, 2011, 25700 km^2, 483 km^3

EUTROPHISATION (III)

Pollution ponctuelle (Ohio)

Drainage de sol et de fertilizant

Plan approximatif

1. Modélisation (physique et mathématique)

2. Identification de sources pour le modèle à un traceur.

3. Pour le modèle couplé à deux traceurs.

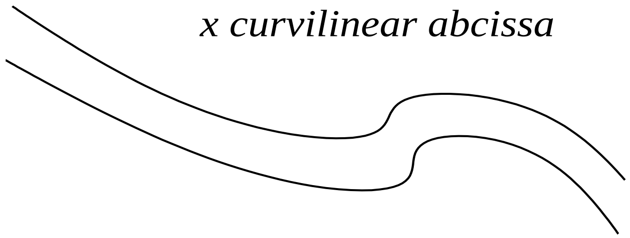
Modélisation?

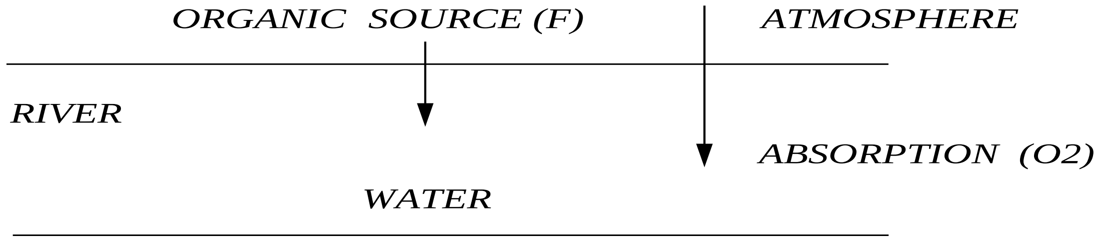
1. Modèle Physique (base de la démarche scientifique)
   On découpe un bout de réalité, compliqué, et on en fait une représentation abstraite à laquelle on va appliquer une théorie.
   Le modèle obtenu n'est pas figé : on peut en faire une première étude d'un modèle simpliste, puis ajouter des paramètres [et] tenter d'obtenir un modèle proche du réel.

2. Modèle Mathématique
   Utiliser des expressions mathématiques pour décrire une situation quantitative réelle [ou proche du réel].
   Consiste à écrire en notation mathématique ce qui est exprimé d'abord en mots en faisant intervenir des variables [et des équations] au besoin.

Traceur DBO

b : Demande Biochimique en Oxygène DBO —BOD en anglais—.





Réaction, auto-épuration (Streeter & Phelps, 1925). $(I = (0, L),\ \partial_x b = b')$

$$\partial_t b = -Rb + F \qquad \text{in } (0, T).$$

Transport, la rivière s'écoule

$$\partial_t b + (Vb)' + Rb = F \qquad \text{in } I \times (0, T),$$

Dispersion : panache

$$\partial_t b - (Db')' + (Vb)' + Rb = F \qquad \text{in } I \times (0, T).$$

Traceur OD

c_O : Concentration de l'Oxygène Dissous

c_S : Déficit de concentration par rapport à la saturation

$c = c_S - c_O$: Déficit d'Oxygène Dissous

Pompe d'Oxygène ($\Longrightarrow$) Déficit d'Oxygène
($\Longrightarrow$) Absorption d'Oxygène de l'atmosphère

(Dispersion, Transport, Réaction)

$$\partial_t c - (Dc')' + (Vc)' + R_* c = G \qquad \text{in } I \times (0, T).$$

BDO-OD : Modèle couplé

Source de Pollution Organique (F), Source (ou puit) d'Oxygène (G)

$$
\begin{aligned}
\partial_t b - (Db')' + (Vb)' + Rb = F \qquad & \text{in } I \times (0,T), \\
\partial_t c - (Dc')' + (Vc)' + R_* c - Rb = G \qquad & \text{in } I \times (0,T), \\
b(0,t) = c(0,t) = 0, \qquad & \text{in } (0,T), \\
Db'(L,t) = Dc'(L,t) = 0, \qquad & \text{in } (0,T), \\
b(0,\cdot) = c(0,\cdot) = 0, \qquad & \text{in } I.
\end{aligned}
$$

Manuels à lire:

Brown (EPA, 1987), Okubo (Springer, 1980).

Large Bibliographie sur le modèle.

Modèle très simplifié, utilisé en 1925

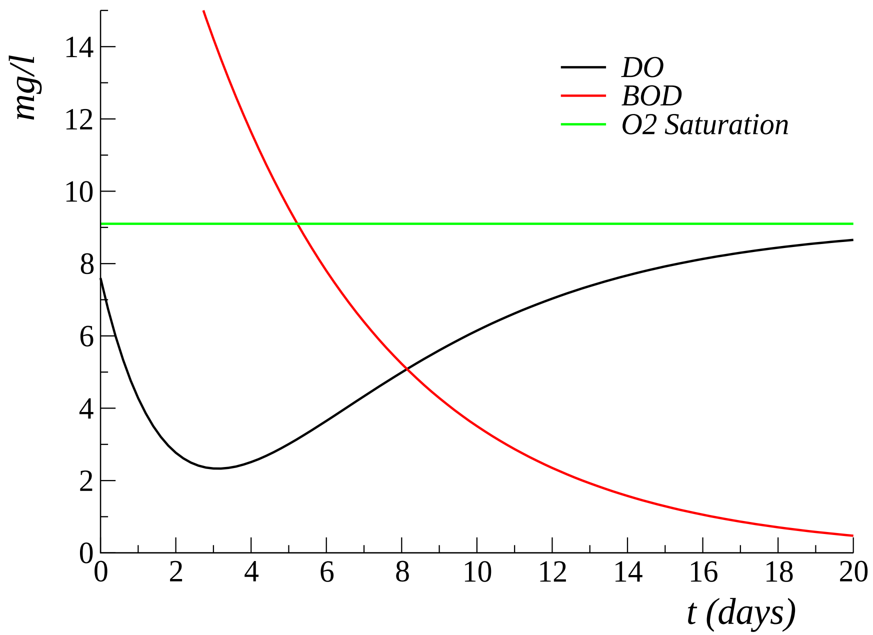
Streeter & Phelps, (Etude de l'Ohio, fleuve nord américain)

$$\begin{aligned} \partial_t b + Rb &= 0 \qquad \text{dans } (0,T), \\ \partial_t c + R_* c - Rb &= 0 \qquad \text{dans } (0,T), \\ b(0) &= b_0 \qquad \text{dans } (0,T). \\ c(0) &= c_0, \qquad \text{dans } (0,T). \end{aligned}$$

Un système d'équations différentielles, . . . ,

Les Terminales S savent résoudre..

Modèle Streeter-Phelps (Illustration)



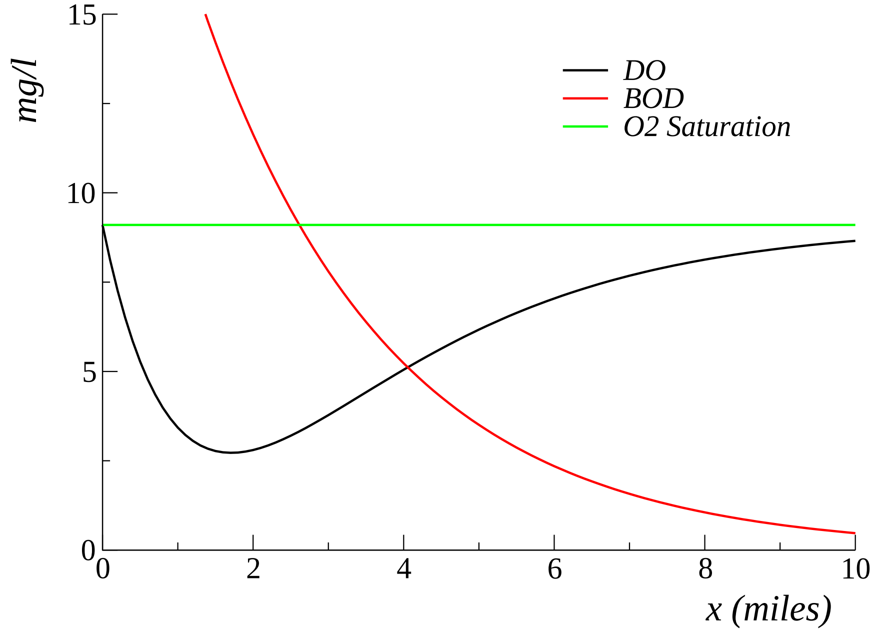


Diagramme gauche : Pollution Initiale, (Auto-épuration)
Diagramme droit : pollution permanente, (Auto-épuration + Mouvement).

Détéction d'une pollution ponctuelle

Sources ponctuelles à identifier

(Source de pollution , puits d'Oxygène)

$$(F(x,t)\ ,\ G(x,t)) = (f(t)\delta_{x-r(t)}\ ,\ g(t)\delta_{x-s(t)}), \qquad \text{in } I \times (0,T).$$

Observations

Observations en temps réel, possible sur c l'O_2 dissous

$$B\,[F,G] = \Big\{(c,Dc')\,(\xi_1,\cdot)\ ,\ (c,Dc')\,(\xi_2,\cdot)\Big\}, \qquad \text{in } (0,T).$$

Hypothèse : Position des observations (amont, aval)

$$\xi_1 < r(t), s(t) < \xi_2, \qquad \text{in } (0,T).$$

Comment utiliser ces observations?

Emploi direct des Observations sur l'OD (c)

Utiliser les observations $(c, Dc')(\xi_1, \cdot), (c, Dc')(\xi_2, \cdot)$ réalisées en temps réel et s'attaquer directement à la detection pour le système couplé DBO-OD.

Observations expérimentalle sur la DBO (b)

Protocole Chimique long $(\Longrightarrow)$ Mesure de $(b(\xi_1, \cdot), b(\xi_2, \cdot))$ sur la DBO

$(\Longrightarrow)$ Détecter la source de pollution sur la DBO, sans utiliser l'OD

$(\Longrightarrow)$ Dure cinq jours! Très Long!

I. Scalar BOD Equation

Adel Hamdi, Mickael Andrle,
Thèses de Mathématiques Appliquées à l'U. T. Compiègne

Switch to english!

BOD Equation

[Advection]-Dispersion- [Reaction] (BOD) Equation

$$\begin{aligned} \partial_t b - (Db')' + [(Vb)'] + [Rb] &= F \qquad && \text{in } I \times (0,T), \\ b(0,t) &= 0, && \text{in } (0,T), \\ Db'(L,t) &= 0, && \text{in } (0,T), \\ b(0,\cdot) &= 0, && \text{in } I. \end{aligned}$$

Steady or Moving Source $(f(\cdot) \not\equiv 0)$

$$F(x,t) = f(t)\delta_{x-r(t)}\ , \qquad \text{in } I \times (0,T).$$

Observations on b

$$B\,[F] = \Big\{ b\,(\xi_1,\cdot)\ ,\ b\,(\xi_2,\cdot) \Big\}, \qquad \text{in } (0,T).$$

Observed Data $\alpha = (\alpha_1\ ,\ \alpha_2)$

$$\text{Find } F; \qquad B\,[F] = \Big\{ \alpha_1\,(\cdot)\ ,\ \alpha_2\,(\cdot) \Big\}, \qquad \text{in } (0,T).$$

IDENTIFIABILITY (IP 2011, PH.D M ANDRLE)

THÉO. **1** *Assume that* $B\,[F_1] = B\,[F_2]$, *then* $F_1 = F_2$ *that is*

$$(r_1(\cdot), f_1(\cdot)) = (r_2(\cdot), f_2(\cdot)), \qquad \textit{in } (0,T).$$

PROOF Simplification : fixed sources (r_1, r_2). Now, define

$$b_1 \;(\Longleftrightarrow)\; F_1, \qquad b_2 \;(\Longleftrightarrow)\; F_2$$

Set $\eta = (b_1 - b_2)$. Assume that $B\,[F_1] = B\,[F_2]$ (with $r_1 < r_2$). Then

$$\eta(\xi_1, \cdot) = \eta(\xi_2, \cdot) = 0, \qquad \text{in } (0,T).$$

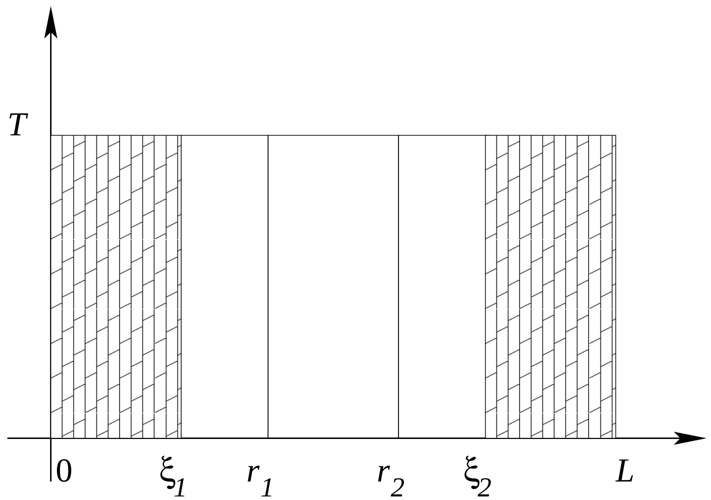
Aim $(\Longrightarrow)$ Show in three steps that

$$\Big(\eta \equiv 0, \qquad \text{in } I \times (0,T)\ \Big)(\Longrightarrow)\ \Big((r_1, f_1(\cdot)) = (r_2, f_2(\cdot)), \qquad \text{in } (0,T)\Big).$$

First Step

Solve heat sub-problems in external strips $(0, \xi_1) \times (0, T)$ (in $(0, \xi_2) \times (0, T)$)

$$\begin{aligned} \partial_t \eta - (D\eta')' &= 0 && \text{in } (0, \xi_1) \times (0, T), \\ \eta(0, t) = \eta(\xi_1, t) &= 0 && \text{in } (0, T), \\ \eta(0, \cdot) &= 0 && \text{in } (0, \xi_1). \end{aligned}$$



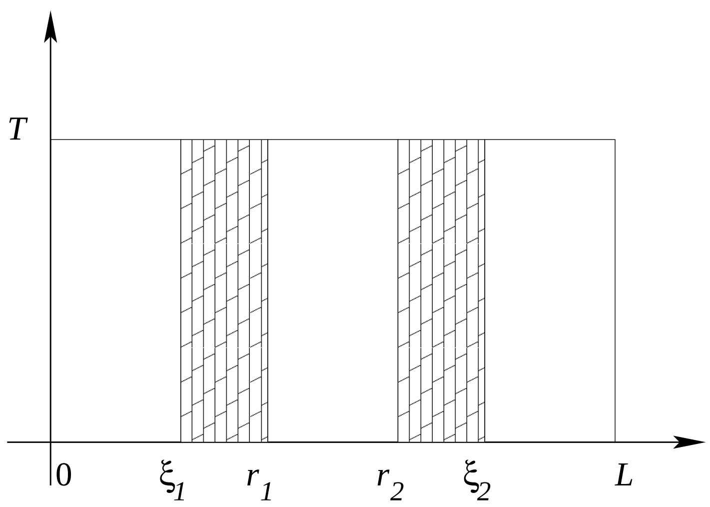
Then, we obtain that

$$\begin{aligned} \eta &\equiv 0, && \text{in } (0, \xi_1) \times (0, T), \\ \eta &\equiv 0, && \text{in } (\xi_2, L) \times (0, T). \end{aligned}$$

Second Step

Solve sideways sub-problems in (ξ_1, r_1) (in (r_2, ξ_2)). Cauchy's conditions at $x = \xi_1$

$$\begin{aligned} \partial_t \eta - (D\eta')' &= 0 && \text{in } (\xi_1, r_1) \times (0, T), \\ \eta(\xi_1, t) = D\eta'(\xi_1, t) &= 0 && \text{in } (0, T), \\ \eta(0, \cdot) &= 0 && \text{in } (\xi_1, r_1). \end{aligned}$$



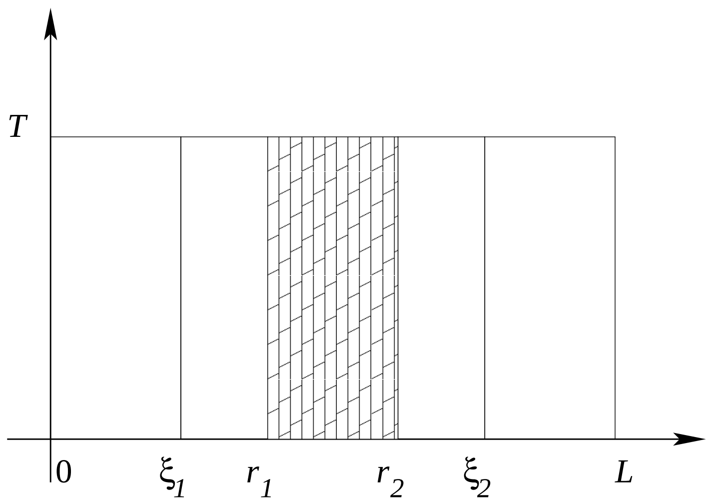
Unique continuation theorem (Saut et Sheurer, 1987),
Carleman estimate (Klibanov, 2006, A. Ben Abdallah et al, 2008-10),
We derive that

$$\eta \equiv 0, \qquad \text{in } (\xi_1, r_1) \cup (r_2, \xi_2) \times (0, T).$$

Third and Final Step

Solve internal sub-problem in (r_1, r_2).

$$\begin{aligned} \partial_t \eta - (D\eta')' &= \ 0 && \text{in } (r_1, r_2) \times (0, T), \\ \eta(r_1, t) = \eta(r_2, t) &= 0, && \text{in } (0, T), \\ \eta(0, \cdot) &= 0, && \text{in } (r_1, r_2). \end{aligned}$$



Then, we have to

$$\eta \equiv 0, \qquad \text{in } (r_1, r_2) \times (0, T).$$

We conclude therefore to

$$\eta \equiv 0, \qquad \text{in } I \times (0, T).$$

The final result is then obtained which is $F_1 = F_2$, or $(f_1(\cdot), r_1(\cdot)) = (f_2(\cdot), r_2(\cdot))$.

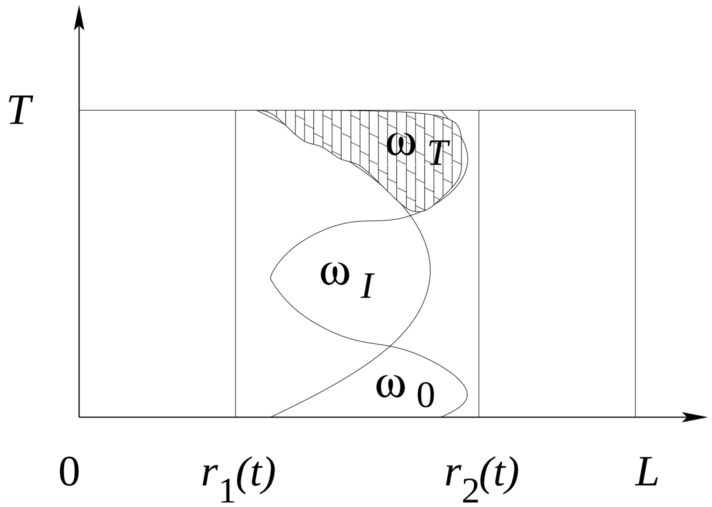
Moving Source : Third and Final Step

The sub-problem is set in non-cylindrical time-space domain, $(\omega_0, \omega_I$ and $\omega_T)$,

$$\begin{aligned} \partial_t \eta - (D\eta')' = \ & 0 \qquad && \text{in } \omega_T, \\ \eta(r_2(t), t) = \eta(r_1(t), t) = \ & 0, \qquad && \text{in } (0, T), \end{aligned}$$

Multiply by η and integrate by parts we derive that

$$\int_{(r_1(T), r_2(T))} \eta(T, x)^2 \, dx + \int_{\omega_T} [D(\eta')^2](t, x) \, dtdx = 0.$$



Then, we have to

$$\eta \equiv 0, \qquad \text{in } \omega_T.$$

We conclude therefore to

$$\eta \equiv 0, \qquad \text{in } I \times (0, T).$$

Numerical Examples (I)

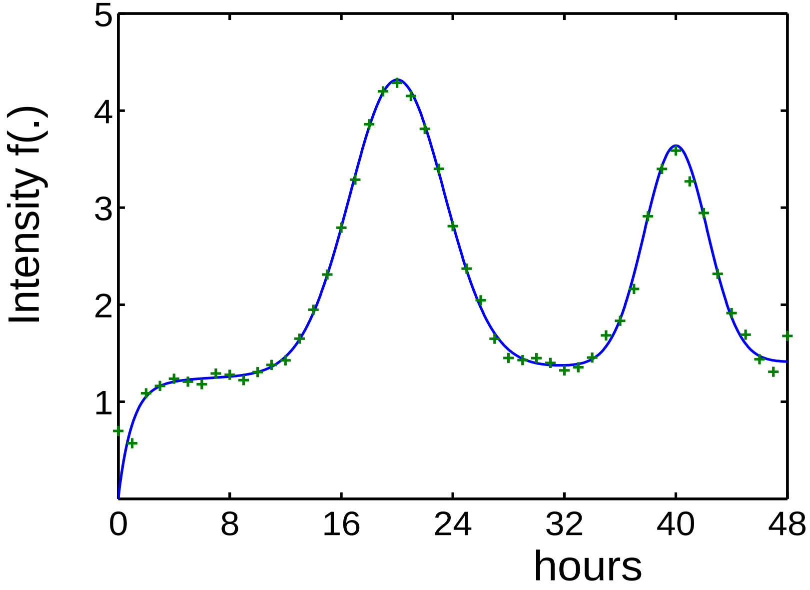
$$L = 1000\ m, \qquad T = 48 \times 3600\ s, \qquad (\xi_1, \xi_2) = (200, 800)\ m$$

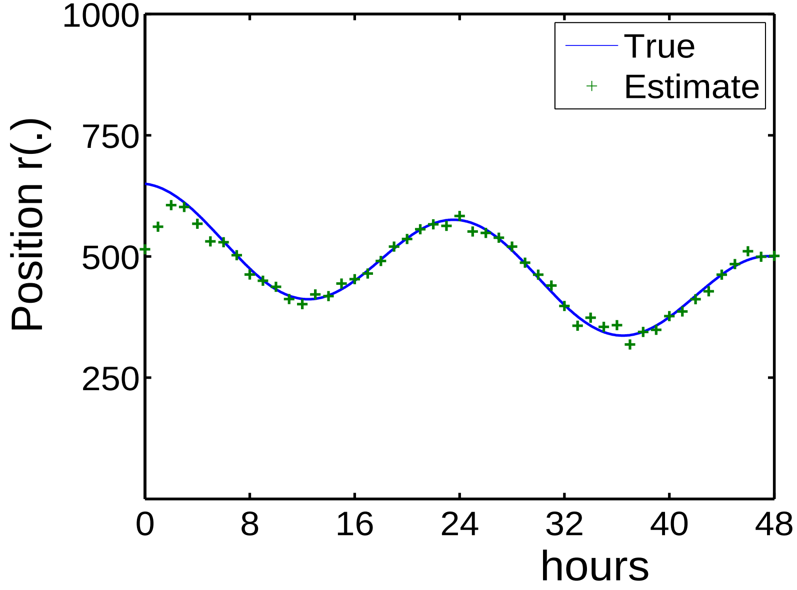
$$D = 40\,\mathrm{m}^2\mathrm{s}^{-1}, \qquad R_* = R = 1 \times 10^{-5}\,\mathrm{s}^{-1}$$

$$r(t) = L\left\{\frac{13}{20} + \frac{1}{10}\left(\cos\left(4\pi\frac{t}{T}\right) - 1\right) - \frac{3t}{20T}\right\}$$

$$f(t) = 3\left\{\exp - \left(\frac{10}{T}\left(t - 7.2 \times 10^4\right)\right)^2 + \frac{3}{4}\exp - \left(\frac{15}{T}\left(t - 1.44 \times 10^5\right)\right)^2 + \frac{2}{5}\left(1 - \exp\frac{-t}{3.6 \times 10^3}\right) + \frac{1}{10}\left(1 - \exp\frac{-t}{1.44 \times 10^5}\right)\right\}$$

Numerical Examples (II)

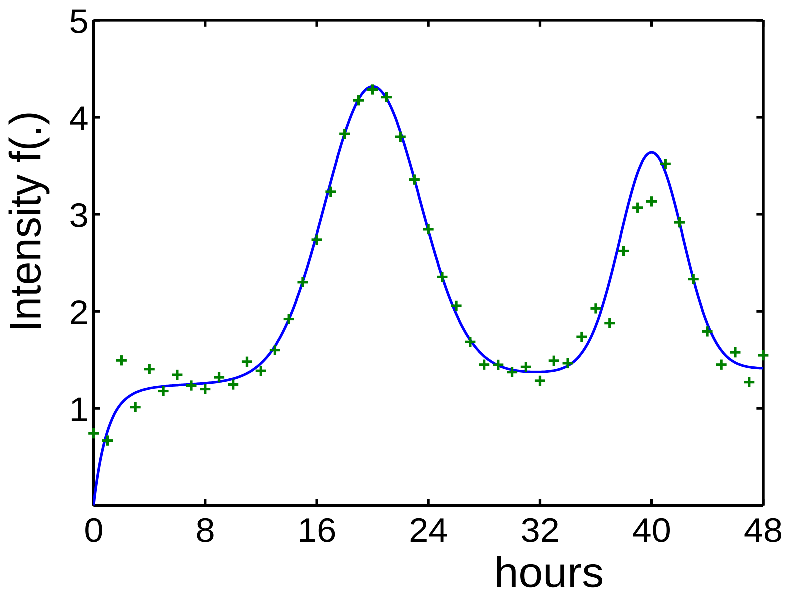


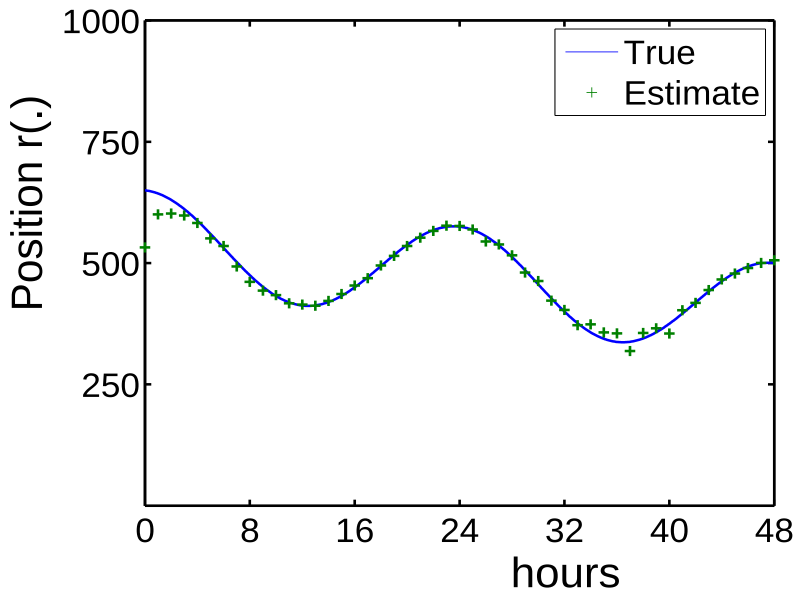


$V = 0.2 \text{ ms}^{-1}$

Noise on Observations 0.05, Errors : Intensity 0.05, Location 0.042

Numerical Examples (III)

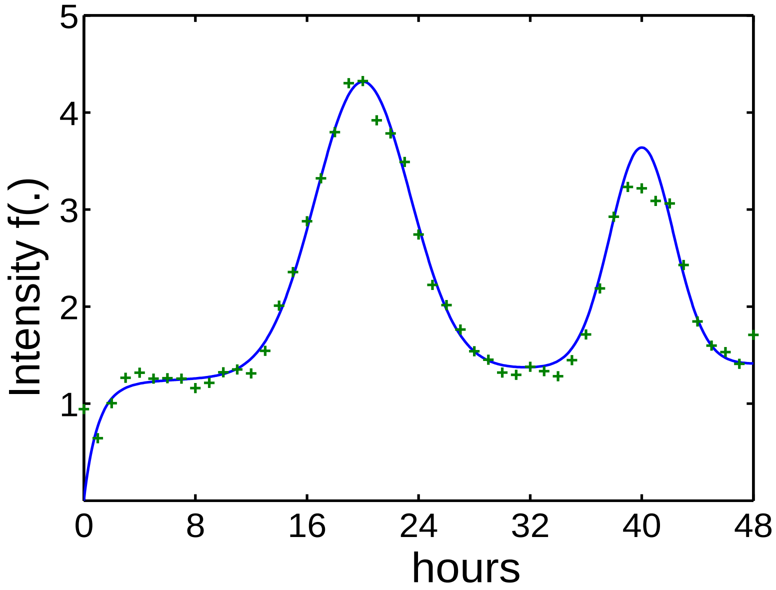


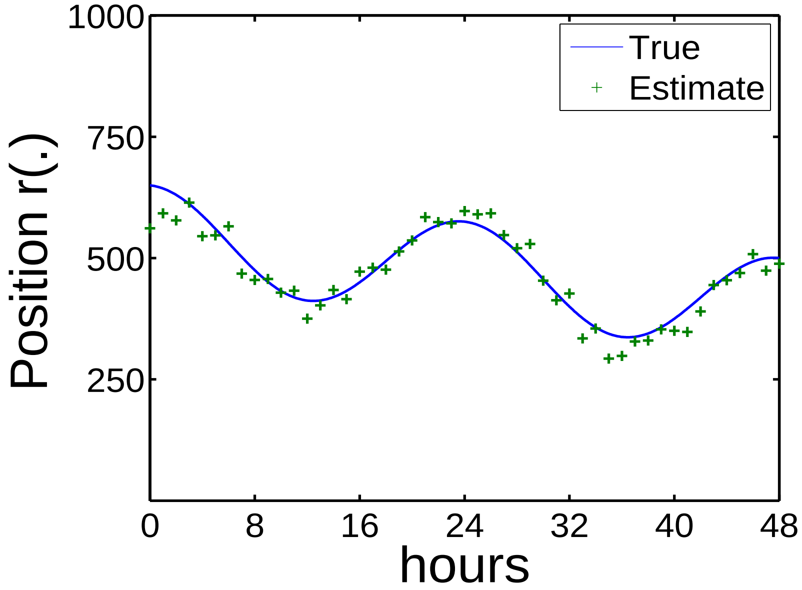


$V = 0.5 \ \ \mathrm{ms}^{-1}$

Noise on Observations 0.05, Errors : Intensity 0.079, Location 0.032

Numerical Examples (IV)





$$V = 0.2 \ \ \mathrm{ms}^{-1}$$

Noise on Observations 0.125, Errors : Intensity 0.062, Location 0.08

II. Coupled BOD-DO Model

Adel Hamdi, Souad Khiari
Thèses de l'Université de Technologie de Compiègne

Sources Detection

(Pollution Source, Oxygen Sink) $\qquad (F, G) = (f(t)\delta_{x-r(t)}, g(t)\delta_{x-s(t)})$

$$
\begin{aligned}
\partial_t b - (Db')' + Rb &= F && \text{in } I \times (0,T),\\
\partial_t c - (Dc')' + R_* c - Rb &= G && \text{in } I \times (0,T),\\
b(0,t) = c(0,t) &= 0, && \text{in } (0,T),\\
Db'(L,t) = Dc'(L,t) &= 0, && \text{in } (0,T),\\
b(0,\cdot) = c(0,\cdot) &= 0, && \text{in } I.
\end{aligned}
$$

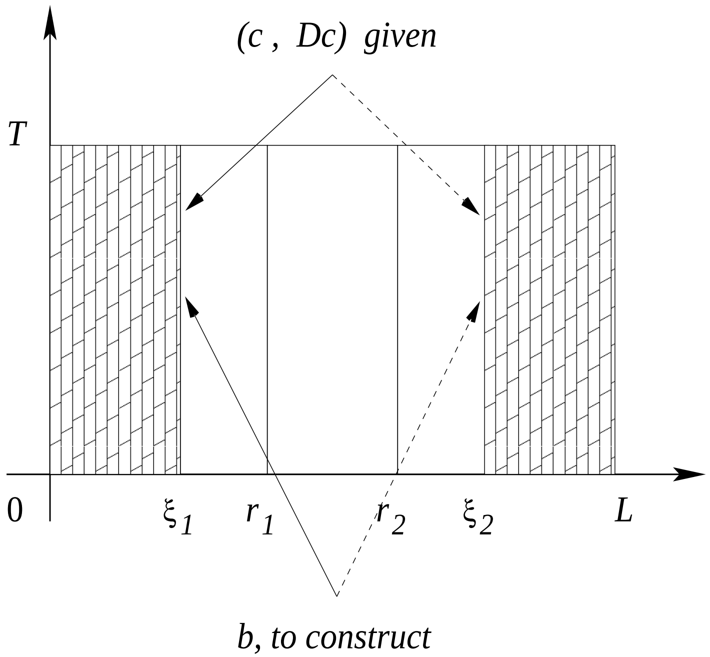
Observations available on the Dissolved Oxygen Concentration c

$$
B\,[F, G] = \Big\{ (c, Dc')\,(\xi_1, \cdot) \;\; , \;\; (c, Dc')\,(\xi_2, \cdot) \Big\}, \qquad \text{in } (0,T).
$$

Observed Data (α, β)

$$
\text{Find } (F, G); \qquad B\,[F, G] = \Big\{ (\alpha, \beta)_1\,(\cdot) \;\; , \;\; (\alpha, \beta)_2\,(\cdot) \Big\}, \qquad \text{in } (0,T).
$$

Methodology



Show that the operators are injective

$$(c(\xi_1,\cdot), Dc'(\xi_1,\cdot)) \mapsto b\,(\xi_1,\cdot)\,,$$

$$(c(\xi_2,\cdot), Dc'(\xi_2,\cdot)) \mapsto b\,(\xi_2,\cdot)\,.$$

Then, join (Part I) in Step 2 to obtain separately identifiability of F and of G.

Sideways BOD-DO Model

Two sub-problems in the time-space strips $(0, \xi) \times (0, T)$.

$$
\begin{aligned}
\partial_t b - (Db')' + Rb = 0 \qquad & \text{in } (0, \xi) \times (0, T), \\
\partial_t c - (Dc')' + R_* c - Rb = 0 \qquad & \text{in } (0, \xi) \times (0, T), \\
b(0, t) = c(0, t) = 0, \qquad & \text{in } (0, T), \\
(b(0, \cdot), c(0, \cdot)) = (0, 0), \qquad & \text{in } (0, \xi). \\
c(\xi, t) = \alpha(t), \qquad & \text{in } (0, T), \\
Dc'(\xi, t) = \beta(t), \qquad & \text{in } (0, T).
\end{aligned}
$$

Uniqueness is the aim!

Ill-posedness (I)

Set $\beta(t) = 0$. Consider $\gamma(t) = Db'(\xi, t)$ as unknown. Solve sequentially

$$
\begin{aligned}
\partial_t b_\gamma - (Db'_\gamma)' + Rb_\gamma &= 0 && \text{in } (0,\xi) \times (0,T),\\
b_\gamma(0,t) &= 0, && \text{in } (0,T),\\
Db'_\gamma(\xi,t) &= \gamma(t), && \text{in } (0,T),\\
b_\gamma(0,\cdot) &= 0, && \text{in } (0,\xi).
\end{aligned}
$$

$$
\begin{aligned}
\partial_t c_\gamma - (Dc'_\gamma)' + R_* c_\gamma &= Rb_\gamma && \text{in } (0,\xi) \times (0,T),\\
c_\gamma(0,t) &= 0, && \text{in } (0,T),\\
Dc'_\gamma(\xi,t) &= 0, && \text{in } (0,T),\\
c_\gamma(0,\cdot) &= 0, && \text{in } (0,\xi).
\end{aligned}
$$

We have to solve the ill-posed equation on $\gamma \in L^2(0,T)$, that is

$$
S\gamma(t) = c_\gamma(\xi,t) = \alpha(t), \qquad \text{in } (0,T).
$$

Ill-posedness (II)

Fourier Computations ($D = R = R_* = 1$)

$$b_\gamma(x,t) = \frac{2}{\xi}\sum_{k\in\mathbb{N}}\left((-1)^k\int_0^t \gamma(s)e^{-\lambda_k(t-s)}\,ds\right)\sin\left((k+\frac{1}{2})\frac{\pi}{\xi}x\right),$$

$$c_\gamma(x,t) = \frac{2}{\xi}\sum_{k\in\mathbb{N}}\left((-1)^k\int_0^t \gamma(s)(t-s)e^{-\lambda_k(t-s)}\,ds\right)\sin\left((k+\frac{1}{2})\frac{\pi}{\xi}x\right),$$

$$(S\gamma)(t) = \int_0^t K(t-s)\gamma(s)\,ds, \qquad \forall t\in(0,T),$$

$$\lambda_k = \left[(k+\frac{1}{2})\frac{\pi}{\xi}\right]^2 + 1, \qquad K(s) = \frac{2}{\xi}\sum_{k\in\mathbb{N}} s e^{-\lambda_k s}, \qquad \forall s\in(0,T).$$

$K(s) = \mathcal{O}(\sqrt{s})$ ($\Longrightarrow$) Singular values of S decrease like $k^{-3/2}$.

($S\gamma = \alpha$) is a mildly ill-posed Volterra equation.

Why the Mild Ill-posedness?

Observation loc. = Reconstruction l. | Observa. loc. $\neq$ Reconstruc. loc.

$db'(0) = ?$ | $db'(\pi) = 0$

$dc'(0) = 0, \; c(0) = \alpha$ | $dc'(\pi) = 0$

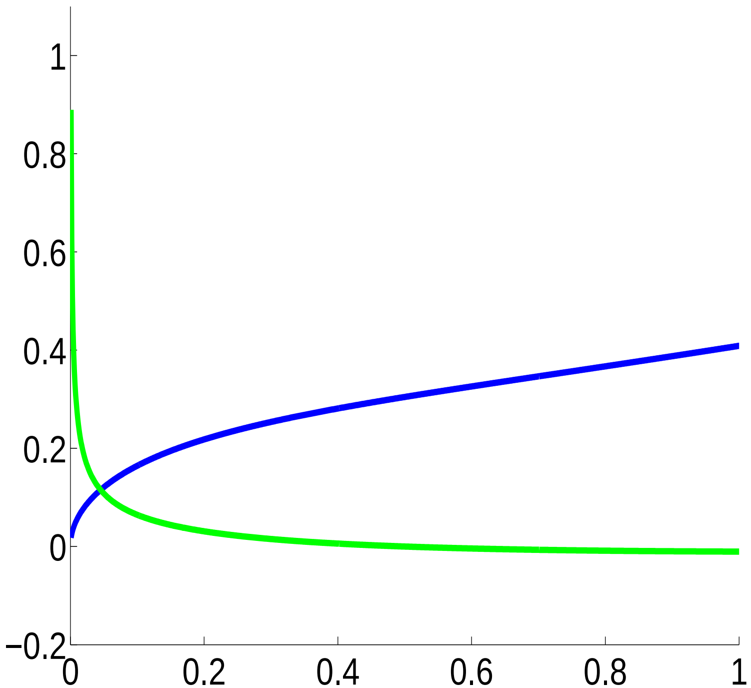
$db'(0) = ?$ | $db'(\pi) = 0$

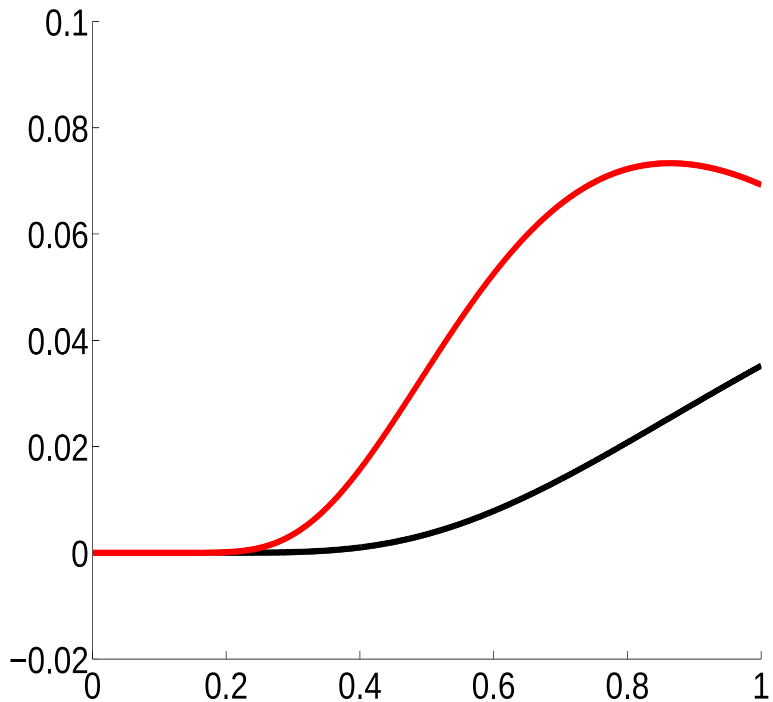
$dc'(0) = 0$ | $dc'(\pi) = 0, \; c(\pi) = \alpha$

Kernels of Volterra Equations (replace 2 by 1 for $k = 0$)

$$K(s) = \frac{1}{\xi} s \sum_{k \in \mathbb{N}} 2e^{-\lambda_k s}, \qquad \tilde{K}(s) = \frac{1}{\xi} s \sum_{k \in \mathbb{N}} 2(-1)^k e^{-\lambda_k s}.$$

Convolution Kernels





Kernels $K(\cdot)$ and $\tilde{K}(\cdot)$ and their first derivatives

!! Different scales in the vertical axis.

Uniqueness : Illustration ($T = \infty$)

Solve equation

$$(S\gamma)(t) = K \star \gamma(t) = 0 \qquad \forall t \in (0, \infty).$$

Use Laplace transform yields that

$$\widehat{K}(p)\hat{\gamma}(p) = 0, \qquad \forall p \in (0, \infty),$$

The Laplace Transform of $K(\cdot)$ is

$$\widehat{K}(p) = \frac{2}{\xi} \sum_{k \in \mathbb{N}} \frac{1}{(p + \lambda_k)^2} > 0, \qquad \forall p \in (0, \infty).$$

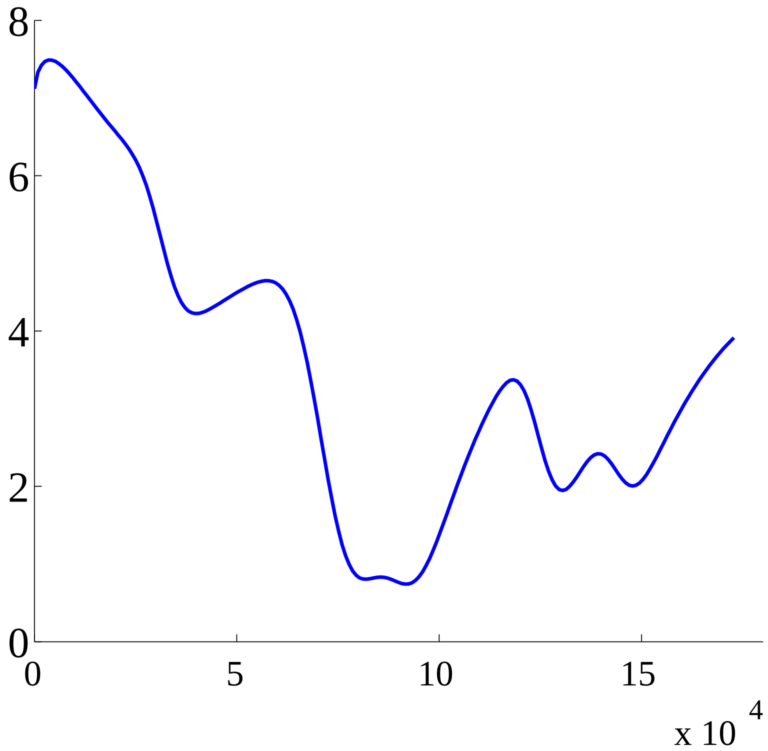
We obtain then that

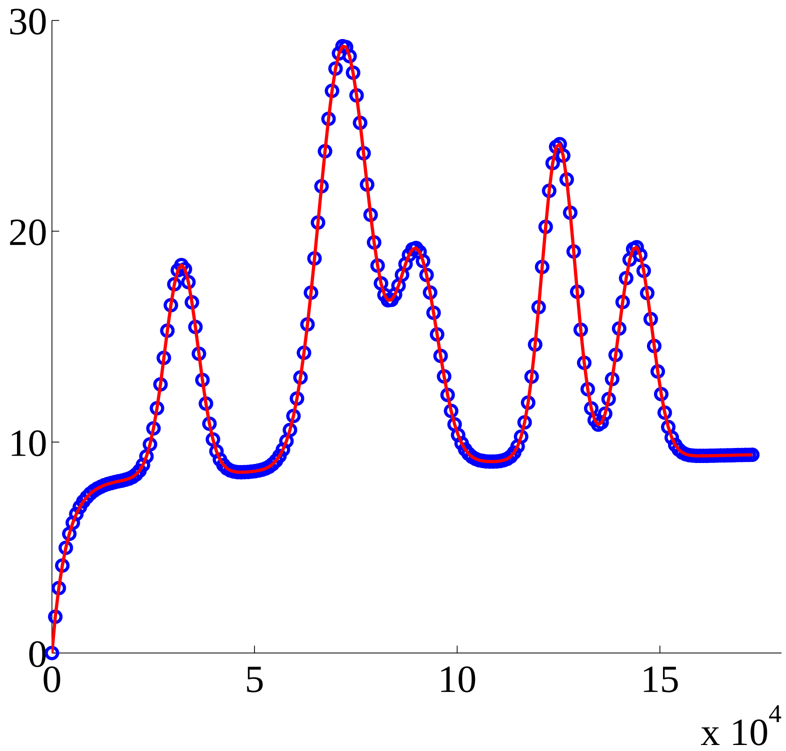
$$(\hat{\gamma}(p) = 0, \ \forall p \in (0, \infty)) \implies (\gamma(t) = 0, \ \forall t \in (0, \infty)).$$

Then

$$b(t, x) = c(t, x) = 0, \qquad \forall t \in (0, \infty), \ \forall x \in (0, L).$$

NUMERICS (I)

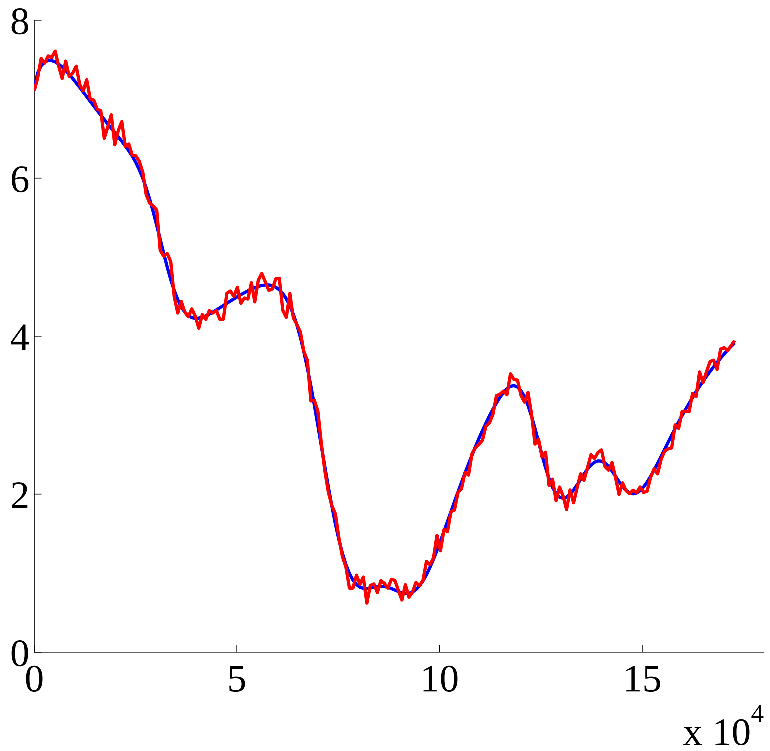


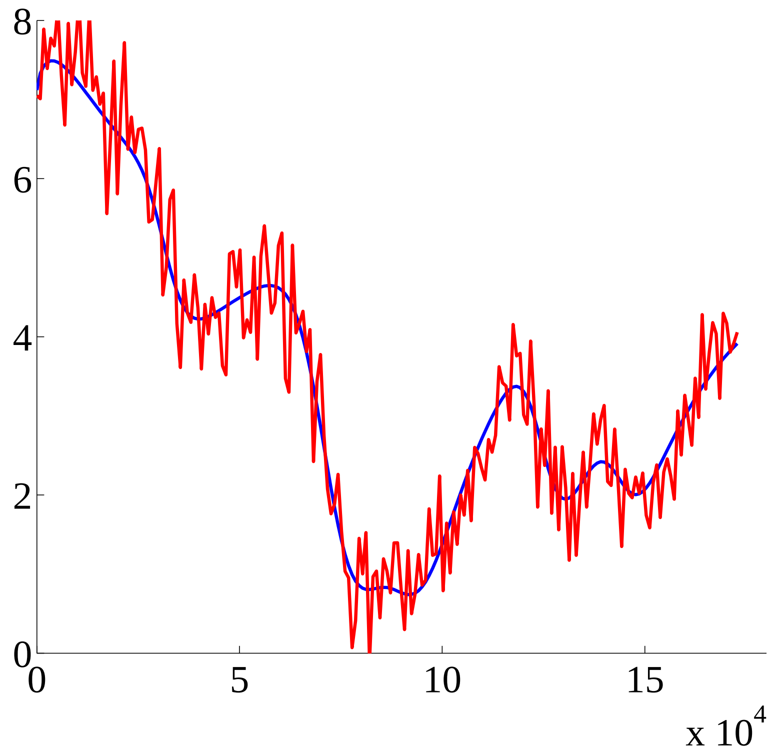


Exact Observation $\alpha(\cdot)$.

Exact BOD load, $b(x = 0, t)$.

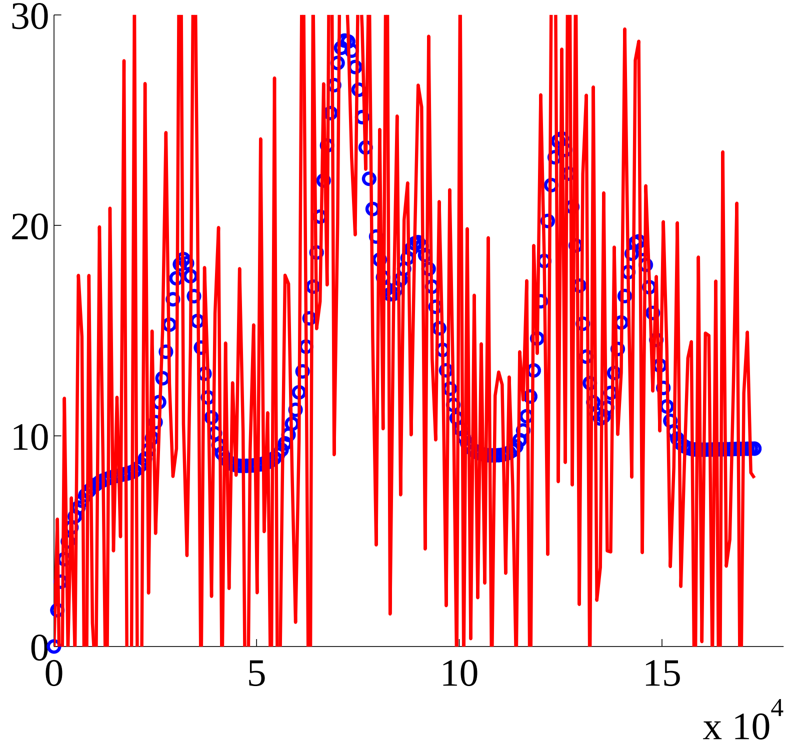
Numerics (II)

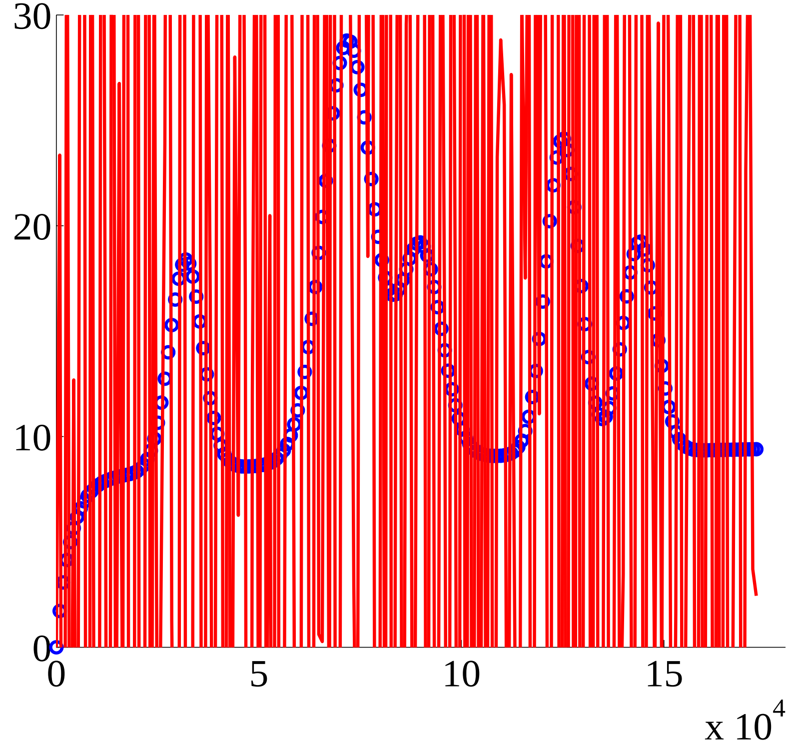




Noisy profiles (Gaussian white noise, deviation $\sigma = 0.1$ and 0.5).

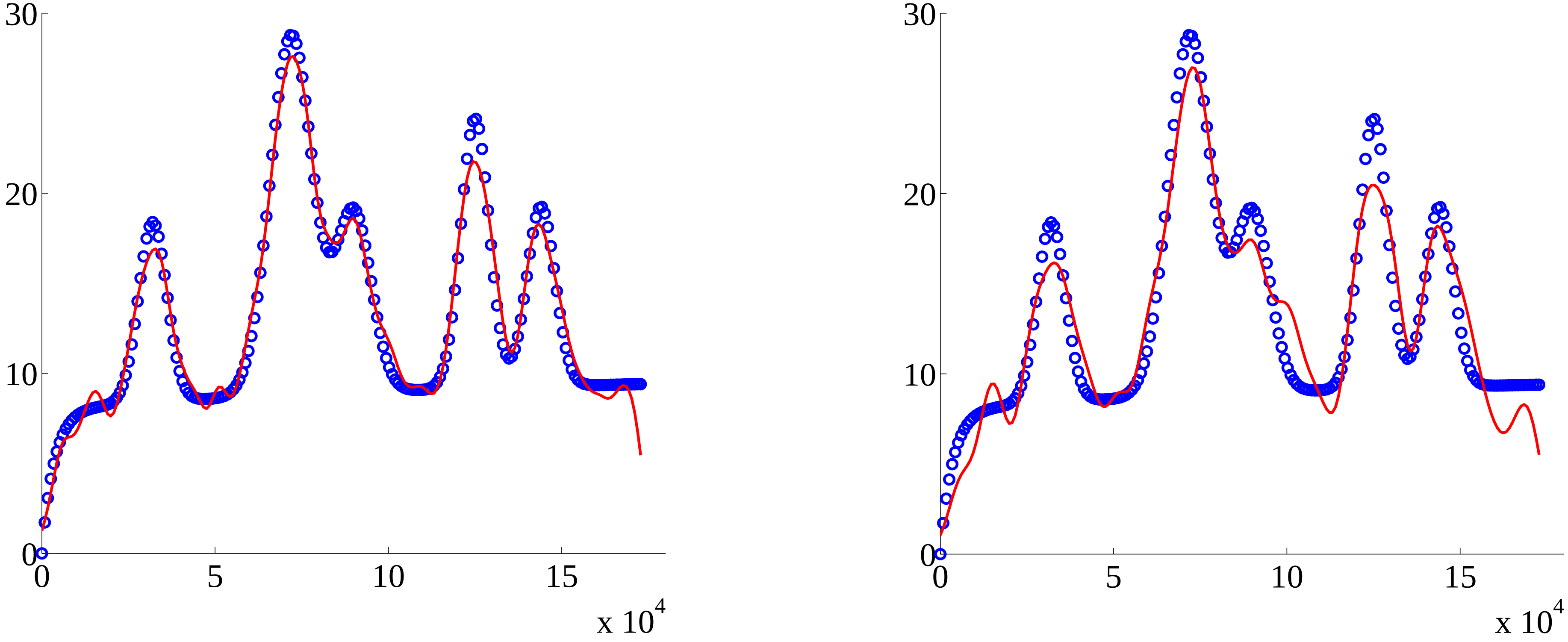
Numerics (III)





Crudely computed DBO loads (Instability).

NUMERICS (IV)



Aposteririo filtered DBO loads (Gauss filter).

General Case : Abstract EDO Formulation

Set $I_\xi = (0, \xi)$. Assume $(f, g) \in L^2(I_\xi \times (0, T))$. Consider the system

$$\partial_t \begin{pmatrix} b \\ c \end{pmatrix} + A \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

The operator A is defined as follows

$$A \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -(Db')' + Rb \\ -(Dc')' + R_* c - Rc \end{pmatrix}$$

Its domain is given by

$$D(A) = \Big\{(b, c) \in \boldsymbol{H}^1(I_\xi), \qquad ((Db')', (Dc')') \in \boldsymbol{L}^2(I_\xi)$$
$$(b(0), c(0)) = (0, 0), \;\; c(\xi) = (Dc')(\xi) = 0\Big\}.$$

Analysis of A

Assume $(f,g) \in L^2(I_\xi)$. Consider the stationary system

$$A\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

Hence

$$\begin{aligned} -(Db')' + Rb &= f \qquad && \text{in } I_\xi, \\ -(Dc')' + R_* c - Rb &= g \qquad && \text{in } I_\xi, \\ b(0) = c(0) &= 0, \\ c(\xi) = 0, \qquad Dc'(\xi) &= 0, \end{aligned}$$

Functional framework

$$b \in V = \Big\{\varphi \in H^1(I_\xi) \qquad \varphi(0) = 0\Big\}, \qquad v \in Q = H_0^1(I_\xi).$$

Saddle-Point Problem

Find $(b, c) \in V \times Q$ such that

$$\int_{I_\xi} Db'\psi'\,dx + \int_{I_\xi} Rb\psi\,dx = \int_{I_\xi} f\psi\,dx, \qquad \forall\psi \in Q,$$

$$\int_{I_\xi} Dc'\varphi'\,dx + \int_{I_\xi} R_* c\varphi\,dx - \int_{I_\xi} Rb\varphi\,dx = \int_{I_\xi} g\varphi\,dx, \qquad \forall\varphi \in V.$$

This is a saddle point problem

$$m(b, \psi) = \int_{I_\xi} f\psi\,dx, \qquad \forall\psi \in Q,$$

$$-a(b, \varphi) + m_*(\varphi, c) = \int_{I_\xi} g\varphi\,dx, \qquad \forall\varphi \in V.$$

Saddle Point Theory (Nicolaides, 1982), (Bernardi, Canuto, Maday, 1988).

Well Posedness of the Steady Problem

Define the Kernel of $m_{(*)}(\cdot,\cdot)$

$$\mathcal{N}\left(m_{(*)}\right) = \left\{\varphi \in V, \qquad m_{(*)}\left(\varphi,\psi\right) = 0 \quad \forall \psi \in Q\right\}.$$

The bilinear form $a(\cdot,\cdot)$ satisfies a two inf-sup conditions on $\mathcal{N}(m) \times \mathcal{N}(m_*)$,

$$\inf_{\psi\in\mathcal{N}(m)} \sup_{\varphi\in\mathcal{N}(m_*)} \frac{a(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geq \eta, \qquad \inf_{\psi\in\mathcal{N}(m_*)} \sup_{\varphi\in\mathcal{N}(m)} \frac{a(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geq \eta_*.$$

The bilinear form $m_{(*)}(\cdot,\cdot)$ satisfies the inf-sup condition in $V \times Q$,

$$\inf_{\psi\in Q} \sup_{\varphi\in V} \frac{m_{(*)}(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geq 1.$$

Prop. **2** *The mixed problem has a unique solution* $(b,c) \in V \times Q$ *such that*

$$\|b\|_{H^1} + \|c\|_{H^1} \leq C(\|f\|_{L^2} + \|g\|_{L^2}).$$

The Resolvent of A

Prop. 3 *Let* $(f,g) \in \boldsymbol{L}^2(I_\xi)$*. Set*

$$(b_\lambda, c_\lambda) = \mathcal{R}(\lambda, A)(f,g) = (\lambda + A)^{-1}(f,g).$$

We have that

$$\|b_\lambda\|_{L^2} + \lambda\|c_\lambda\|_{L^2} \leq C(\|f\|_{L^2} + \|g\|_{L^2}),$$

The resolvent $\mathcal{R}(\lambda)$ *is then a bounded in* $\boldsymbol{L}^2(I_\xi)$

$$\|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I)\to\boldsymbol{L}^2(I_\xi))} \leq C'$$

Rem. 1 *We have not*

$$\cancel{\|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi)\to\boldsymbol{L}^2(I_\xi))} \leq \frac{C}{\lambda}}.$$

Hille-Yosida Theory fails (Expected!).

Uniqueness (I) : Pazy's Theorem

Théo. **4 (A. Pazy)** *If* $\mathcal{R}(\lambda)$ *exists for large real-numbers* $\lambda(\geq 0)$ *and*

$$\limsup_{\lambda\to+\infty} \frac{1}{\lambda} \log \|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi)\to\boldsymbol{L}^2(I_\xi))} = 0,$$

then the ODE problem has at most one solution.

Rem. 2 *The norm of the resolvent may behave like*

$$\|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi)\to\boldsymbol{L}^2(I_\xi))} = \exp(\lambda\epsilon(\lambda)).$$

A Uniqueness Result for the Sideways Problem

THÉO. **5** *The Sideways Problem has at most one solution.*

PROOF We have that $\mathcal{R}(\lambda)$ is well defined for all $\lambda \geq 0$. The uniform resolvent estimate yields that

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi)\to\boldsymbol{L}^2(I_\xi))} = 0.$$

Applying Pazy's Theorem completes the proof.

Identifiability for the sources in BOD-DO

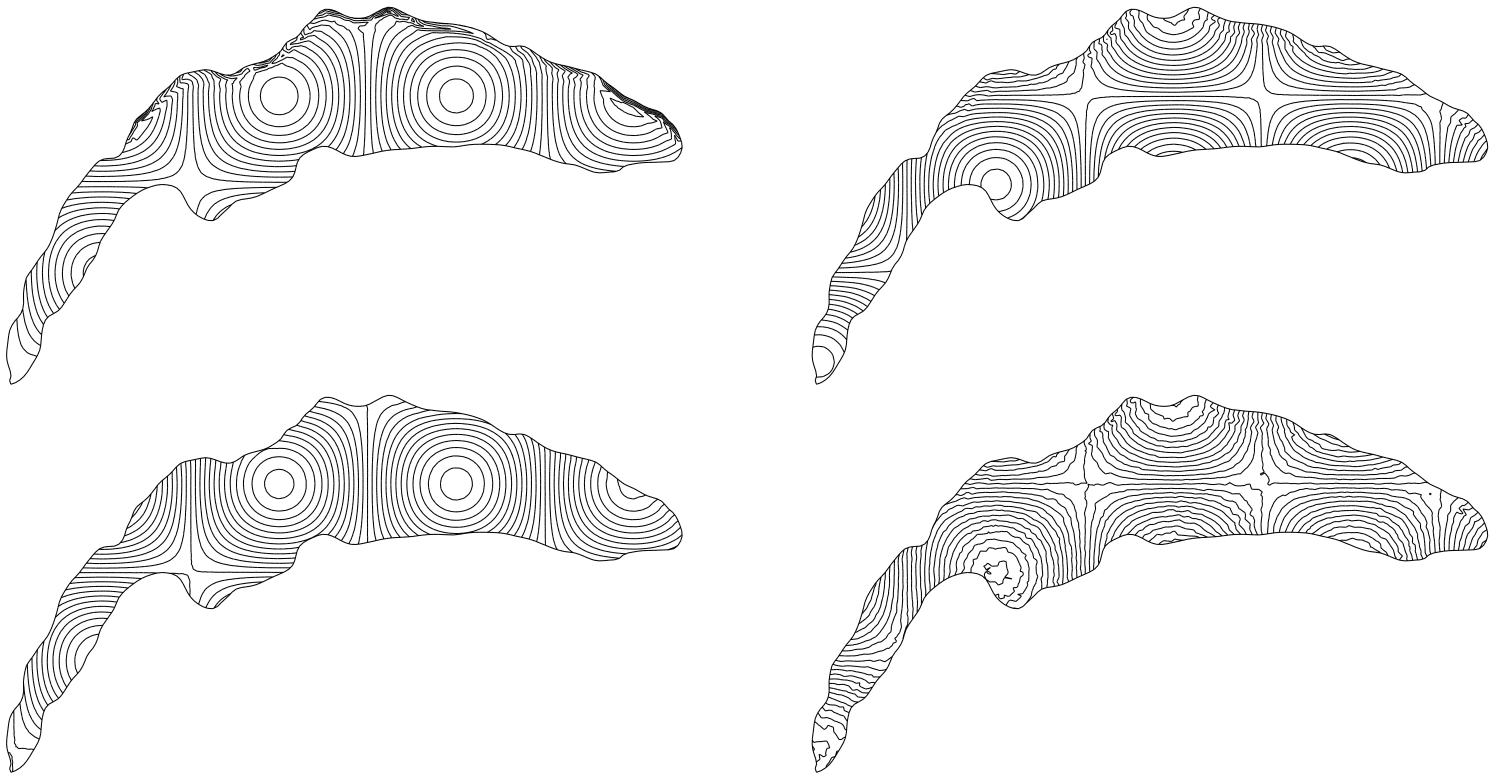
Théo. 6 *Assume that* $B[F_1, G_1] = B[F_2, G_2]$, *then*

$$(r_1(\cdot), f_1(\cdot)) = (r_2(\cdot), f_2(\cdot)), \qquad (s_1(\cdot), g_1(\cdot)) = (s_2(\cdot), g_2(\cdot)), \qquad \textit{in } (0, T).$$

mD Extension

- Space Finite Element Discretization of the BOD/DO data completion
  1. Steady problem : Not easy technically! Only partially solved.
     SINUM , 2015 —Compiègne Paris Reims—-
  2. UnSteady problem :
     FE Semi-discretization ($\Longrightarrow$) Differential Algebraic Equation (DAE).
     ($\Longrightarrow$) More knowledge about existence Issue, Instability, ...
     J. Inverse & Ill-posed Problems, accepted
     S. Khiari, PhD.

- Source detection for full BOD/DO model, 1D first and for mD later.
  M. Andrle a commencé, voir Inverse Poblems, Volume 28, 2012



The unsteady model. Computations are made within FREEFEM$_{++}$.